WHAT IS THE AREA OF A SURFACE?

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1. The usual formula. Let $x=x(u)$, $y=y(u)$, $z=z(u)$ be a parametric representation of a curve C, where x, y, z are Cartesian coördinates.* If we assume that $x(u)$, $y(u)$, $z(u)$ have continuous derivatives of the first order in an interval $u_1 \leqq u \leqq u_2$, except possibly at a finite number of points, then the length $l(C)$ of C is given by the formula

$$l(C) = \int_{u_1}^{u_2} \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 \right]^{1/2} du. \tag{1}$$

Similarly, if $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $u_1 \leqq u \leqq u_2$, $v_1 \leqq v \leqq v_2$, is a parametric representation of a surface S, and if $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of the first order, except possibly along a finite number of smooth arcs, then the area $A(S)$ of S is given by the formula

$$A(S) = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \left[\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2 \right]^{1/2} du dv. \tag{2}$$

2. Objections. For the average mathematician, who has no occasion to consider surfaces more general than those described above, the answer to the question, stated in the title, is very simple: the area of a surface is the value furnished by formula (2). While this answer is entirely satisfactory as far as it goes, it is open to at least two fundamental objections.

To illustrate the first objection, let us consider the formula $V=4\pi abc/3$ which gives the volume of an ellipsoid in terms of its axes. For most practical purposes, this formula may be taken as the *definition* of the volume of an ellipsoid, but a mathematician would surely insist that this formula represents a *theorem*. Similarly, we shall insist that the formulas (1) and (2) should be proved, on the basis of previously stated definitions of length and area.

A second fundamental objection is based on the observation that more and more mathematicians are being concerned with very general curves and surfaces, especially in Calculus of Variations. For somewhat general curves and surfaces the formulas (1) and (2) are not as reliable as may be desired. Let us consider the following example. Starting with the well-known Cantor function, it is easy to construct a function $f(u)$ with the following properties. a) $f(u)$ is continuous and strictly increasing in $0 \leqq u \leqq 1$. b) $f'(u)=0$ in $0 \leqq u \leqq 1$, with the exception of a set of measure zero. c) $f(0)=0$, $f(1)=1$. The equations $x=f(u)$, $y=f(u)$, $z=f(u)$, $0 \leqq u \leqq 1$, describe then the motion of a particle that starts at (0, 0, 0), moves continuously and steadily ahead along the line $x=y=z$, and arrives ultimately at the point (1, 1, 1). Thus the length of the path of this particle is $\sqrt{3}$. On the other hand, formula (1) yields, for this same path, the value zero (of

* We are actually interested in *surfaces*. References to curves serve the purpose of illustrating the analogies and discrepancies between arc-length and area.

course, the integral in (1) is now taken in the Lebesgue sense). A similar example may be constructed for surfaces.

In view of these remarks, we need, first of all, proper definitions for length and area. If C is a continuous curve, then the definition of its length $l(C)$ may be described by the formula

$$l(C) = \lim l(\Gamma_n), \tag{3}$$

where Γ_n designates any sequence of inscribed polygons, with maximum side-lengths converging to zero. This definition agrees with our intuition as to the way to measure length. To justify (3) mathematically, it should be shown that the limit in (3) exists and is independent of the particular choice of the sequence Γ_n. This can be done quite easily. It should be observed that for certain continuous curves C the length may be infinite, as simple examples will show.

3. Schwarz's example. In view of the striking resemblance between formulas (1) and (2), mathematicians were quite surprised, presumably, when about eighty years ago H. A. Schwarz called attention to the following example. Let S designate the lateral surface of a right circular cyclinder, with altitude 1, and with a base circle whose radius is also equal to 1. Cut S along a generator and then spread it upon a plane. We obtain a rectangle R with sides 1 and 2π. Thus the area of S is equal to 2π. Subdivide the sides of R into m and n equal parts respectively and draw lines, parallel to the sides, through the points of division. R is thus subdivided into mn small congruent rectangles. We subdivide each of these rectangles into four triangles by drawing diagonals. Let us now bend R so as to obtain the original cyclindrical surface S. We obtain then, on S, a set of $4mn$ curvilinear triangles, in each of which we replace the curvilinear sides by their respective chords. The result is an inscribed polyhedron $\Sigma_{m,n}$ with $4mn$ rectilinear triangular faces. An elementary computation yields the following formula for the area of $\Sigma_{m,n}$:

$$A(\Sigma_{m,n}) = 2n \sin\frac{\pi}{2n} + \left[\frac{1}{4} + \frac{4m^2}{n^4}\left(n \sin\frac{\pi}{2n}\right)^4\right]^{1/2} \times 2n \sin\frac{\pi}{n}.$$

Let us choose $m = n^3$ and let us denote by Σ_n the inscribed polyhedron $\Sigma_{n^3,n}$. It follows that $A(\Sigma_n)$ converges, for $n \to \infty$, to infinity, and not to $2\pi = A(S)$. The interested reader will easily determine the totality of the limits that can be obtained by properly coördinating the values of m and n.

It is now clear that the areas of inscribed polyhedra, approximating a given surface, will not converge, in general, to the area of that surface.

The example of Schwarz, discussed in the preceding section, was the starting point of an extensive and fascinating literature. Still, we do not possess as yet a satisfactory *theory* of the area of surfaces, even though the subject attracted the attention of many mathematicians. Historical evidence seems to indicate that progress in this theory is closely dependent upon the stimulating influence of the demands placed upon it by current research in other fields, especially in Calculus of Variations. It is probable that the recent sustained interest in two-

dimensional problems in Calculus of Variations will lead, in a not very distant future, to decisive developments in the theory of the area of surfaces.

4. Lebesgue's definition. Even the briefest survey of the present status of the theory of the area would exceed, by far, the limitations of space that we have to observe on this occasion.* However, we shall attempt to give some information of interest to the non-specialist, who may now insist, in view of the example of Schwarz, that we show him some valid definition of the area of a surface. We shall state presently such a definition, due to Lebesgue. This definition, probably the most fruitful one of the many definitions proposed to date, may be derived, by analogy, from an alternative form of the definition of the arc-length which we shall describe first.

Let there be given continuous curves C: $x=x_n(u)$, $y=y_n(u)$, $z=z_n(u)$, $u_1 \leqq u \leqq u_2$, $n=0, 1, 2, \cdots$. Suppose $C_n \to C_0$ in the sense that $x_n(u) \to x_0(u)$, $y_n(u) \to y_0(u)$, $z_n(u) \to z_0(u)$ uniformly in $u_1 \leqq u \leqq u_2$. It is then easily proved that

$$l(C_0) \leqq \liminf l(C_n). \tag{4}$$

This relation expresses the fundamental fact that the arc-length is a *lower semi-continuous functional*. In view of the fundamental importance of such functionals in various fields, it may be desirable to possess an alternative definition of the arc-length that is based upon its property of being lower semi-continuous. An elementary reasoning yields the formula

$$l(C) = \text{gr.l.b. } \liminf l(\Gamma_n), \tag{5}$$

where the greatest lower bound is taken with respect to all sequences of polygons Γ_n (*not necessarily inscribed*), such that $\Gamma_n \to C$ in the sense explained earlier in this section.

We may now take (5) as a *definition* of the arc-length (of course (3) becomes a *theorem*). If we replace in the definition described by (5), the terms *length*, *polygon*, *curve* by the terms *area*, *polyhedron*, *surface* respectively, then we obtain the definition of Lebesgue for the area of a surface.

If we adopt the definition (5) for the arc-length and the corresponding definition of Lebesgue for the area of a surface, then we have re-established the analogy that we had a right to expect in view of the obvious resemblance of formulas (1) and (2). Of course, we should now *prove* formulas (1) and (2), at least for the curves and surfaces described in section 1. For the curves of section 1 the proof is quite simple. The literature on the Lebesgue area contains a proof of formula (2) under extremely general (although probably not final) assumptions. It may not be an easy task to derive from these proofs a presentation that would account for the surfaces of section 1 and at the same time be suitable for classroom use.

* The following publications of the writer may be used as a start in exploring the literature. a) On the problem of Plateau Ergebnisse der Mathematik, vol. 2. b) On the semi-continuity of double integrals in parametric form, Transactions of the American Mathematical Society, vol. 51, 1942, pp. 336–361.

THE SOLUTION OF p-ADIC EQUATIONS

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1. Introduction. In a paper* presented for the Slaught Memorial Volume of this MONTHLY, C. C. MacDuffee has given a brief expository discussion of Hensel's p-adic numbers. In particular, consideration was given to the solution in the field Ω_p of the equation $f(x)=0$, where $f(x)$ has rational integral coefficients. Assuming the existence of a root of the form

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots, \qquad 0 \leqq a_i < p, \tag{1}$$

we consider this root to be known as soon as a process is determined whereby as many of the coefficients a_i as we please may be successively found. The leading coefficient a_0 having been found as a solution of the congruence $f(x) \equiv 0 \pmod{p}$, a technique was shown for solving the equation provided $f'(a_0) \not\equiv 0 \pmod{p}$. We shall refer to this as the regular case. However, if $f'(a_0) \equiv 0 \pmod{p}$, the method not only fails to yield a root α, but leaves in doubt even the existence of such a root.

The present paper develops a different method which enables us to ascertain in a finite number of steps how many roots the equation has in Ω_p, as well as to solve for any root α whose existence has been demonstrated. Throughout the paper it will be assumed that $f(x)$ is of the form

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0,$$

since otherwise it could be reduced to this form by a transformation $x' = kx$.

2. The regular case. In order to compare the two techniques, we shall present, with slight modifications, MacDuffee's proof of

THEOREM 1. *The equation $f(x)=0$ will have a solution in Ω_p if $f(x) \equiv 0 \pmod{p}$ has a solution $x=a_0$ such that $f'(a_0) \not\equiv 0 \pmod{p}$.*

Let us assume that $f(x)=0$ has a solution α of the form (1), and set

$$\alpha_0 = a_0, \ \alpha_1 = a_0 + a_1 p, \ \alpha_2 = a_0 + a_1 p + a_2 p^2, \cdots,$$
$$\alpha_k = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.$$

Then $\alpha \equiv \alpha_k \pmod{p^{k+1}}$, whence $f(\alpha) \equiv f(\alpha_k) \pmod{p^{k+1}}$. Since $f(\alpha)=0$,† this implies that $f(\alpha_k) \equiv 0 \pmod{p^{k+1}}$; for $k=0$ we have

$$f(\alpha_0) \equiv 0 \pmod{p}, \tag{2}$$

which is therefore a necessary condition for a solution.

Since $\alpha_{k+1} = \alpha_k + a_{k+1}p^{k+1}$, we shall have

* C. C. MacDuffee, The p-adic numbers of Hensel, this MONTHLY, vol. 45, No. 8, pp. 500–508.

† This means that $\phi[f(\alpha)]=0$, where ϕ represents the valuation defined for the p-adic field. This paper employs the valuation defined in MacDuffee's paper: if $a = p^r\gamma$ where $(\gamma, p)=1$, then $\phi(a)=p^{-r}$.

$$\begin{aligned} f(\alpha_{k+1}) &= f(\alpha_k) + f'(\alpha_k)a_{k+1}p^{k+1} + \cdots \\ &\equiv f(\alpha_k) + f'(\alpha_k)a_{k+1}p^{k+1} \pmod{p^{k+2}} \\ &\equiv 0 \pmod{p^{k+2}}. \end{aligned}$$

Since $f(\alpha_k) \equiv 0 \pmod{p^{k+1}}$, there is an integer h_k such that

$$f(\alpha_k) = h_k p^{k+1}. \tag{3}$$

Then

$$h_k p^{k+1} + f'(\alpha_k)a_{k+1}p^{k+1} \equiv 0 \pmod{p^{k+2}},$$

whence

$$f'(\alpha_k)a_{k+1} + h_k \equiv 0 \pmod{p}. \tag{4}$$

Now $\alpha_k = a_0 + pg$ where g is a rational integer. Then

$$\begin{aligned} f'(\alpha_k) &= f'(a_0) + f''(a_0)pg + \cdots, \\ &\equiv f'(a_0) \pmod{p}, \end{aligned}$$

and from (4) we have

$$f'(a_0)a_{k+1} + h_k \equiv 0 \pmod{p}. \tag{5}$$

If $f'(a_0) \not\equiv 0 \pmod{p}$, (5) may be solved for a_{k+1}. A solution of $f(x) = 0$ can then be obtained by successively solving (3) for h_k and (5) for a_{k+1} when $k = 0, 1, 2, \cdots$. That a number

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$$

found by this procedure is actually a root of $f(x) = 0$, may be seen immediately. For, since $f(\alpha) - f(\alpha_k) \equiv 0 \pmod{p^{k+1}}$, then

$$\begin{aligned} \phi[f(\alpha) - f(\alpha_k)] &\leqq p^{-k-1} \\ &< \frac{\epsilon}{2} \end{aligned}$$

for k sufficiently large. Similarly, since $f(\alpha_k) \equiv 0 \pmod{p^{k+1}}$,

$$\phi[f(\alpha_k)] < \frac{\epsilon}{2}.$$

Then

$$\phi[f(\alpha)] < \epsilon$$

and $f(\alpha) = 0$.

3. Another method of solution. The method of the preceding section makes use solely of the given equation $f(x) = 0$ and may therefore be considered analogous to Newton's method for approximating an irrational root of an al-

gebraic equation. The analogy is further preserved in the condition imposed upon $f'(a_0)$. We shall here develop a technique comparable to Horner's method, the successive approximations being obtained by means of a chain of equations

$$f(x) = 0,\ F_1(x) = 0,\ F_2(x) = 0, \cdots, F_i(x) = 0,$$

the equation $F_i(x)=0$ yielding the coefficient a_i.

If $f(x)=0$ has a solution in Ω_p of the form $\alpha=a_0+\alpha_1 p$ where $\alpha_1=a_1+a_2 p+a_3 p^2+\cdots$, then

$$f(a_0+\alpha_1 p) = f(a_0) + f'(a_0)\alpha_1 p + \cdots + \alpha_1^n p^n = 0.$$

Let $f(a_0)=k_0 p$. Then

$$p[k_0 + f'(a_0)\alpha_1 + \cdots + \alpha_1^n p^{n-1}] = 0,$$

so that α_1 is a solution of the equation

$$F_1(x) = k_0 + f'(a_0)x + \cdots + x^n p^{n-1} = 0.$$

The solution of $F_1(x)=0$ in Ω_p is in all respects like that of $f(x)=0$. We may thus obtain a sequence of functions $F_i(x)$, each derived from the preceding in exactly the same manner that $F_1(x)$ was derived from $f(x)$. Successive approximations to the solution of $f(x)=0$ may be obtained from the chain of equations without returning each time to a consideration of the original equation. Thus if $\alpha=a_0+a_1 p+a_2 p^2+\cdots$ is a solution of $f(x)=0$ in Ω_p, then $\alpha_1=a_1+a_2 p+\cdots=a_1+\alpha_2 p$ is a solution of $F_1(x)=0$, and in general $\alpha_i=a_i+\alpha_{i+1}p$ is a solution of $F_i(x)=0$.

Since

$$f(\alpha) = pF_1(\alpha_1) = p^2F_2(\alpha_2) = \cdots = p^kF_k(\alpha_k),$$

it follows that any p-adic number α determined by the above procedure will be a solution of $f(x)=0$, since

$$\phi[f(\alpha)] \leqq p^{-k} < \epsilon$$

for sufficiently large values of k.

In solving the equation $F_1(x)=0$, it is, of course, permissible to divide out a power of p which may be a common factor of all the coefficients. That is always the case when $f'(a_0)\equiv 0 \pmod p$ if the given equation has a solution. For $F_1(x)\equiv k_0+f'(a_0)x \pmod p$, and if $f'(a_0)\equiv 0 \pmod p$ while $k_0\not\equiv 0$, then $F_1(x)\equiv 0 \pmod p$ has no solution. But if p divides k_0, then $F_1(x)$ is divisible by a power of p, which may be divided out and the quotient designated as $F_1(x)$. Similarly we may simplify any of the $F_i(x)$.

4. Recurrence of the functions $F_i(x)$. Two problems present themselves for consideration: (a) the possibility of a recurrence $F_i=F_j$* for all $i>j$, (b) the possi-

* Any recurrence of the form $F_{j+k}=F_j$ for $k>1$ would imply a non-integral rational root. Such a root cannot exist since the leading coefficient of $f(x)$ is unity.

bility that for every i we may have $f(a_0) \equiv f'(a_0) \equiv 0$, $F_i(a_i) \equiv F_i'(a_i) \equiv 0 \pmod{p}$. The latter is of considerable significance, for, although the method of the preceding section did not require the existence of an i such that $F_i'(a_i) \not\equiv 0 \pmod{p}$, we know by Theorem 1 that if such an i exists there will be a corresponding root of $F_i(x) = 0$ and hence of $f(x) = 0$. Such an i having been found, we may determine subsequent coefficients a_k as in Theorem 1. Problem (b) will be discussed in the next section. We shall find the answer to problem (a) in the following theorem and its corollary.

THEOREM 2. *A necessary and sufficient condition that $F_1(x) = f(x)$ is that either $f(x) = (x+1)^n$ or $f(x) = x^n$.*

If $f(a_0) \equiv 0 \pmod{p}$, we have

$$f(a_0 + px) = f(a_0) + f'(a_0)px + \cdots + p^n x^n. \tag{6}$$

$F_1(x)$ being defined to be the quotient after all powers of p are divided out of the right member of (6), if $F_1(x) = f(x)$ the factor p^n must be common to every term and hence $f(a_0 + px) = p^n F_1(x)$ identically. Since

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

on equating coefficients of equal powers of x we obtain

$$c_{n-1} = \frac{na_0}{p-1}$$

and, in general,

$$c_{n-k} = \binom{n}{k}\left(\frac{c_{n-1}}{n}\right)^k = \binom{n}{k}\left(\frac{a_0}{p-1}\right)^k.$$

Hence $f(x) = (x + a_0/p - 1)$ and, since c_{n-k} is an integer, it follows that a_0 is either 0 or $p-1$.

The sufficiency of the condition follows immediately on forming $F_1(x)$ when $f(x)$ is x^n or $(x+1)^n$.

COROLLARY. *A necessary and sufficient condition that $F_i(x) = F_j(x)$ for every $i > j$ is that $F_j(x) = (x+1)^n$ or x^n.*

That the condition given in the corollary is exceptional is indicated in the following theorems.

THEOREM 3. *A necessary and sufficient condition that there exist a value of j such that $F_j(x) = (x+1)^n$ is that $f(x) = (x+a)^n$ where $a > 0$.*

THEOREM 4. *A necessary and sufficient condition that there exist a value of j such that $F_j(x) = x^n$ is that $f(x) = (x-a)^n$ where $a > 0$.*

We shall first prove the sufficiency of the condition of Theorem 3. Let a be written in the form $a = k_0 + k_1 p + \cdots + k_r p^r$ where $0 \leqq k_i < p$, and k_m is the first

non-zero coefficient. We may successively show that for $0 \leqq i < m$ we have $a_i = 0$ and, for $i < m$,

$$F_i(x) = (x + k_m p^{m-i} + k_{m+1} p^{m-i+1} + \cdots + k_r p^{r-i})^n;$$

for $m \leqq i < r$ we have $a_i = p - k_i$ and

$$F_i(x) = (x + 1 + k_i + k_{i+1} p + \cdots + k_r p^{r-i})^n,$$

whence $F_r(x) \equiv (x+1+k_r)^n$ and $F_{r+1}(x) = (x+1)^n$.

Conversely, let $F_{r+1}(x) = (x+1)^n$, it being assumed that the leader $A_r = a_0 + a_1 p + \cdots + a_r p^r$ of the root α has been found by successively forming $F_i(x)$ for $i \leqq r$. It will be shown that $f(x) = (x+a)^n$ where $a = p^{r+1} - A_r$.

We have

$$\begin{aligned} F_r(a_r + px) &= F_r(a_r) + F_r'(a_r) px + \cdots + p^n x^n \\ &= p^n F_{r+1}(x) \\ &= p^n (x^n + n x^{n-1} + \cdots + 1). \end{aligned}$$

On equating coefficients of like powers of x, we obtain

$$F_r(a_r) = p^n, \; F_r'(a_r) = n p^{n-1}, \cdots, F_r^{(n)}(a_r) = n!.$$

Let $F_r(x) = x^n + b_1 x^{n-1} + \cdots + b_n$. After successive differentiation and substitution of the above values, we obtain a system of linear equations,

$$\frac{n!}{(n-k)!} a_r^{n-k} + \frac{(n-1)!}{(n-k-1)!} b_1 a_r^{n-k-1} + \cdots + k! b_{n-k} = \frac{n!}{(n-k)!} p^{n-k},$$

$$(k = 0, 1, 2, \cdots, n-1).$$

The matrix of this system is non-singular and the system has the unique solution $b_s = \binom{n}{s}(p - a_r)^s$. Hence $F_r(x) = (x + p - a_r)^n$.

Now, in exactly the same way it may be shown that if $F_i(x) = (x+c)^n$, then $F_{i-1}(x) = (x + cp - a_{i-1})^n$. It follows that

$$\begin{aligned} F_{r-1}(x) &= [x + (p - a_r)p - a_{r-1}]^n \\ &= (x + p^2 - a_r p - a_{r-1})^n, \\ F_{r-2}(x) &= (x + p^3 - a_r p^2 - a_{r-1} p - a_{r-2})^n, \end{aligned}$$

and finally

$$\begin{aligned} F_0(x) = f(x) &= (x + p^{r+1} - a_r p^r - a_{r-1} p^{r-1} - \cdots - a_0)^n \\ &= (x + a)^n. \end{aligned}$$

The proof of Theorem 4 is in every respect like that of Theorem 3 and will be omitted.

5. Multiple roots.* It is clear that, if $f(x)=0$ has a multiple root α in Ω_p, then α is a root of $f'(x)=0$ in that field. Let $\alpha=a_0+\alpha_1 p$. Then $\alpha_i=a_i+\alpha_{i+1}p$ will be a multiple root of $F_i(x)=0$ and hence a root of $F_i'(x)=0$ for all finite values of i. However, it is easily determined if $f(x)=0$ has a multiple root in Ω_p, since the euclidean algorithm for the greatest common divisor can be applied to the polynomials $f(x)$ and $f'(x)$. If multiple roots occur the degree of the equation may be reduced. We shall therefore assume that $f(x)=0$ has no multiple roots in Ω_p.

Let us assume that

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$$

is a root of $f(x)=0$ in Ω_p such that $f(a_0)\equiv f'(a_0)\equiv 0$, and such that for every finite value of i, $F_i(a_i)\equiv F_i'(a_i)\equiv 0 \pmod{p}$.

It will be true that $f(a_0+a_1p)\equiv 0 \pmod{p^2}$. Now for every integer x,

$$f(a_0 + xp) \equiv f(a_0) + f'(a_0)xp \pmod{p^2}.$$

Since $f'(a_0)\equiv 0 \pmod{p}$, it follows that $f(a_0)\equiv 0 \pmod{p^2}$. That is,

$$f(a_0 + xp) \equiv 0 \pmod{p^2}.$$

Now if we define $F_1(x)$ by the equation

$$f(a_0 + xp) = p^{\beta_1}F_1(x)$$

where β_1 is maximal, it follows that $\beta_1 \geqq 2$. Then

$$f'(a_0 + xp) = p^{\beta_1-1}F_1'(x),$$
$$f'(a_0 + a_1p) = p^{\beta_1-1}F_1'(a_1).$$

But we have assumed that $F_1'(a_1)\equiv 0 \pmod{p}$, so

$$f'(a_0 + a_1p) \equiv 0 \pmod{p^{\beta_1}}.$$

Thus a_0+a_1p is a multiple root of $f(x)\equiv 0 \pmod{p^{\beta_1}}$.

If we define $F_2(x)$ by the equation

$$F_1(a_1 + xp) = p^{\beta_2}F_2(x)$$

where β_2 is maximal, it may be shown by similar reasoning that $\beta_2 \geqq 2$. Then

$$f(a_0 + a_1p + xp^2) = p^{\beta_1}F_1(a_1 + xp) = p^{\beta_1+\beta_2}F_2(x).$$

Hence

$$f'(a_0 + a_1p + xp^2) = p^{\beta_1+\beta_2-2}F_2'(x),$$
$$f'(a_0 + a_1p + a_2p^2) = p^{\beta_1+\beta_2-2}F_2'(a_2).$$

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* The author is indebted to Professor MacDuffee for suggestions in the development of this section.

But we have assumed that $F_2'(a_2) \equiv 0 \pmod{p}$, so

$$f'(a_0 + a_1 p + a_2 p^2) \equiv 0 \pmod{p^{\beta_1+\beta_2-1}}.$$

Thus $a_0+a_1p+a_2p^2$ is a multiple root of $f(x) \equiv 0 \pmod{p^{\beta_1+\beta_2-1}}$.

By induction it can be shown that $a_0+a_1p+\cdots+a_ip^i$ is a multiple root of $f(x) \equiv 0 \pmod{p^s}$ where

$$s = \beta_1 + \beta_2 + \cdots + \beta_i - 1, \qquad \beta_i \geqq 2.$$

Since $s \geqq i+1$, it follows that α is a multiple root of $f(x)=0$ in Ω_p, contrary to our assumption.

It therefore follows that if all multiple roots of $f(x)=0$ have been removed, there must be a first value of i, say $i=m$, such that either $F_m(x) \equiv 0 \pmod{p}$ has no solution, or else $F_m(a_m) \equiv 0$ and $F_m'(a_m) \not\equiv 0 \pmod{p}$. In the former case $f(x)=0$ has no solution with the leader $a_0+a_1p+\cdots+a_ip^i$. In the latter case the congruence $F_m(x) \equiv 0 \pmod{p}$ has a root of lower multiplicity than the preceding congruences $F_i(x) \equiv 0 \pmod{p}$. Continuing, a congruence $F_k(x) \equiv 0 \pmod{p}$ is finally obtained whose roots are distinct and such that $F_k'(a_k) \not\equiv 0 \pmod{p}$. The root can therefore be found to an arbitrary number of places. We have proved

THEOREM 5. *If $f(x)=0$ has only simple roots in Ω_p, it is possible to determine in a finite number of steps, how many solutions there are in Ω_p.*

6. Examples. (a) The equation $f(x)=x^3+43x^2+159x+1217=0$ has no solution in Ω_7, for although the congruence $f(x) \equiv 0 \pmod 7$ has the solution $a_0=2$, we find that $F_1(x)=x^3+49x^2+49x+275 \equiv x^3+2 \equiv 0 \pmod 7$ has no solution.

(b) In the solution of $f(x)=x^3+3x^2+9x+338=0$ in Ω_3 we first find the unique solution $a_0=1$ of $f(x) \equiv 0 \pmod 3$. Then $F_1(x)=x^3+2x^2+2x+13$, and $F_1(x) \equiv 0 \pmod 3$ has two solutions, $a_1=2$ and $a_1=1$. Since $F_1'(2) \not\equiv 0 \pmod 3$ there is at least one solution corresponding to $a_1=2$ and this can be shown to be unique. Proceeding with $a_1=1$, we find $F_2(x)=3x^3+5x^2+3x+2$ and, since $F_2 \equiv 0 \pmod 3$ has no solution, $f(x)=0$ has in Ω_3 but one solution, namely $\alpha=1+2\cdot 3+1\cdot 3^2+2\cdot 3^3+0\cdot 3^4+\cdots$.

(c) In solving $f(x)=x^3+3x^2+300x+5590=0$ in Ω_3 we find successively $a_0=2$, $F_1=x^3+3x^2+36x+230$, $a_1=1$, and $F_2=x^3+2x^2+5x+10$. Now the congruence $F_2 \equiv 0 \pmod 3$ has the two solutions $a_2=1$ and $a_2=2$ and, since $F_2'(2) \not\equiv 0 \pmod 3$, there is a corresponding solution. Proceeding with $a_2=1$, we find $F_3=3x^3+5x^2+4x+2$, $a_3=2$ and $F_4=9x^3+23x^2+20x+6$. The congruence $F_4 \equiv 0 \pmod 3$ has two solutions, $a_4=0$ and $a_4=2$. Since $F_4(0) \not\equiv 0 \pmod 3$ and $F_4(2) \not\equiv 0 \pmod 3$, there is a solution corresponding to each of these values of a_4. Thus $f(x)=0$ has in Ω_3 the three solutions

$$\alpha = 2 + 1\cdot 3 + 1\cdot 3^2 + 2\cdot 3^3 + 0\cdot 3^4 + \cdots,$$
$$\beta = 2 + 1\cdot 3 + 1\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + \cdots,$$
$$\gamma = 2 + 1\cdot 3 + 2\cdot 3^2 + 0\cdot 3^3 + 2\cdot 3^4 + \cdots.$$

CERTAIN MATHEMATICAL ACHIEVEMENTS OF JAMES GREGORY

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For a long time the light of James Gregory did not shine as brightly as did that of John Wallis, Isaac Barrow and Isaac Newton, the other three great British mathematicians of the seventeenth century. Only recently, through the endeavours of several Scottish mathematicians, especially E. T. Whittaker, G. A. Gibson and H. W. Turnbull, Gregory's genius is revealed and fills with admiration all those interested in the development of modern mathematics.

The "*James Gregory Tercentenary Memorial Volume*," edited by H. W. Turnbull [1], contains Gregory's momentous scientific correspondence, mostly with J. Collins. An extremely important supplement is the large number of Gregory's hitherto unpublished notes, recording his mathematical ideas and calculations. These notes were found in a collection of documents in the University of St. Andrews Library, written on the blank spaces of letters to Gregory. This material affords the possibility of studying his achievements and ideas.

In this paper we shall discuss Gregory's expansions of general and particular functions into series. In addition, we shall exhibit the ideas which are set forth in his first mathematical publication "*Vera circuli et hyperbolae quadratura*" [2]. These ideas are concerned, to some extent, with the associated problem of constructing by certain limiting processes the functions which measure the areas of circles and conics.

1. The "Taylor's series". In a letter of February 15, 1671 to J. Collins (see "*Memorial*" [1], pp. 170 ff.) Gregory gives the power series for seven important functions, each with 5 or 6 terms. These functions are, if for the sake of brevity we may use modern notations,

$$\text{arc tan } x,\ \tan x,\ \sec x,\ \log \sec x,\ \log \tan\left(\frac{x}{2}+\frac{\pi}{4}\right),$$

$$\text{arc sec } (\sqrt{2}\,e^{x}),\qquad 2 \text{ arc tan}\left(\tanh\frac{x}{2}\right).$$

He mentions without further explanation that he had some knowledge of Newton's "universal method." Hereby, he refers to some series which Newton had discovered and which Collins had but recently communicated to him.

We may surmise that he obtained the arc tangent series in a way analogous to that by which three years earlier N. Mercator [3] had found the series for $\log(1+x)$. He may have considered arc tan x as the area under the curve $y=(1+x^2)^{-1}$, transformed $(1+x^2)^{-1}$ by formal division into a power series and finally integrated this infinite sum. However, there is no possibility of obtaining the other series in a similar way.

On the blank space of a letter to Gregory, dated January 29, 1671, Turnbull found a group of computations about just these seven functions [4]. The com-

parison of these computations with Gregory's expansions indicates the way of his thoughts. First, they include almost without exception, as many of the successive derivatives of the functions, as would be needed in finding the 5 or 6 numerical coefficients of the series by successive differentiation. Second, all coefficients in Gregory's series are correct with the exception of a single coefficient in both the expansions for $\tan x$ and for $\log \sec x$. (The second error is a consequence of the first since he obviously obtained the log sec series by integrating the tangent series.) Finally, all derivatives in Gregory's notes are correct with the exception of a single numerical error in the derivatives of $\tan x$, which was probably due to miscopying one number. However, using this erroneous value one finds exactly the erroneous coefficients in the series for $\tan x$ and $\log \sec x$. From these two facts, Turnbull argues conclusively that Gregory used the tables of the derivatives for the construction of his power series.

We see two possibilities for such a construction. On the one hand, we may imagine that Gregory applied in each particular case something like the "method of undetermined coefficients" together with successive differentiation. That he mentions "Newtons universal method" immediately before giving his series may be considered as supporting this assumption. In fact, if we look upon the whole of Newton's work we are justified in assuming that Gregory thought of this combined method as "Newtons universal method," even though the idea had been sketched as early as 1637 by Descartes in his "géométrie," and had since been applied by many other mathematicians. Nevertheless, Gregory's remark must be considered as a mere guess based upon the few results from Newton's still unpublished investigations which Collins had communicated to him with no hint about Newton's method.

On the other hand, we may suppose that Gregory could have applied the same process for an unspecified function and could have obtained the general expression for the nth coefficient of the expansion. Thus he would have anticipated Taylor's classical expansion by forty-four years. Neither the letters nor the other material, so far as published, substantiate the latter possibility. From all these facts, we may conclude that Gregory possessed a method for finding the Taylor expansion of any *particular* function, but we cannot affirm that he possessed Taylor's formula for an *unspecified* function.

It may be interesting that the second man, C. Maclaurin, whose name is closely associated with this series, deduced it seventy years later, in his "*Treatise of Fluctions*" (1742) by a reasoning similar to that of Gregory. Of course he applied it at once to an unspecified function. He quotes Taylor's book for the formula but could not have known Gregory's discovery then buried in the correspondence.

2. The interpolation formula. For the independent discovery by Gregory of a famous interpolation formula, full evidence is given in a letter of his published long ago. Nevertheless, nobody seems to have realized this fact until E. T. Whittaker brought it to general notice. In the letter to Collins [5] of November 23, 1670 Gregory stated explicitly a formula which interpolates for a

function $y=f(x)$ when its values at equidistant points $0, c, 2c, 3c, \cdots$ are given. This formula is identical with the famous formula

$$(1)\quad f(x) = f(0) + \frac{x}{c}\Delta f(0) + \frac{x(x-c)}{c\cdot 2c}\Delta^2 f(0) + \frac{x(x-c)(x-2c)}{c\cdot 2c\cdot 3c}\Delta^3 f(0) + \cdots,$$

which Newton made known some years later [6] and which mostly bears his name. It is not essential that Gregory assumes here $f(0)=0$. Further, we may note that, of course, he did not have for the differences the notation $\Delta f(0), \Delta^2 f(0), \Delta^3 f(0), \cdots$. This came into use much later under the influence of Leibniz's symbolism. He takes single letters $d, f, h, \cdots$ for these values, carefully defined by forming the sequences of the 1st, 2nd, 3rd, $\cdots$ differences. Newton uses almost the same notation as Gregory.

In the correspondence on this formula between Collins and Gregory [7], there is mentioned the procedure which Briggs had used in extending his table of logarithms to subintervals. Briggs took differences, generalizing the older method of linear interpolation. His procedure can be considered in some way as the predecessor of the interpolation formula. However, Briggs does not state such a formula nor does he give any motivation of this procedure. Gregory's formula was given in answer to a question raised by Collins for such a motivation.

Of course, Gregory also states his formula without a proper proof, but it is obvious that he could and did verify the formula for polynomials. The same is true for Newton's first publications, although later, in the "*methodus differentialis*," he sketches a way to derive the formula. It is interesting that the interpolation of tables is only *one* aim of Gregory's statement; he emphasizes strongly its use for the problem of approximate quadrature of curves and gives various formulas in this connection. Incidentally Newton [8] makes the same application of the interpolation formula.

The infinite process which is involved in this interpolation formula implies a serious mathematical difficulty which even its discoverers may have felt semi-consciously. The polynomial $P_n(x)$ of the nth degree which is given by the first $n+1$ terms of the formula (1) takes on the values of $f(x)$ at the equidistant points $0, c, 2c, \cdots, nc$, and is determined by this property. This, obviously, is the essential fact which was discovered and communicated by Gregory and Newton. Yet they tacitly assumed that for other unspecified values of x the successive polynomials $P_n(x)$ yield an approximation to $f(x)$ which can be improved by increasing n. Apparently, they thought only of such values of x which are located *between* $0, c, \cdots, nc$, that is to say, they considered only the proper problem of *interpolation*. Here the fact of the steadily improved approximation looks rather evident although a precise formulation and an exact proof was not within the range of these early developments. Things are different if one turns to the problem of *extrapolation*, considering values x *outside* the interval of the multiples of c. The published material gives no evidence that Gregory used his formula for extrapolation. And Newton in the "*Philosophiae Naturalis Prin-*

cipia" [6] applies the interpolation formula not in order to find the place of a comet at any time beyond the range of the observations, but only for intermediate moments.

It is important to realize this situation since the way from the interpolation formula to the Taylor series goes through a sort of extrapolation. Assuming c infinitely small, one concentrates $0, c, 2c, \cdots$ in an arbitrarily small neighborhood of a fixed value and one seeks an expression for $f(x)$ at another fixed value at a finite distance. This can be done formally by applying the usual symbols of the difference and differential calculus. One has only to replace, corresponding to this limiting process, the nth difference quotient $\Delta^n y/\Delta x^n$ in Newton's formula by the nth derivative $d^n y/dx^n$. But in doing so one leaps over a very serious difficulty, using the symbols without regard to their original meaning. In fact, the higher derivatives are defined originally by iteration of the differentiation process (limit of first difference quotient) and their connection with the higher difference quotients is not trivial. And still more difficult for a critical mathematician is the whole limiting process from the interpolation formula to the infinite series. Perhaps such difficulties make us understand why Gregory did not state any connection between his two great results and why Newton, so far as we know, never formulated the Taylor series.

The first to dare to leap over these gaps was Brook Taylor in 1715 [9]. He could do so, since he obviously knew not only Newton's methods but also the concepts and notations introduced in the mean time by Leibniz. He did not use the symbols of Leibniz, but, adapting them to Newton's language, he developed a notation of his own which may, of course, appear a little awkward to us. He applied this symbolism without being influenced by the intrinsic difficulties mentioned above. Thus he came automatically from the interpolation formula to his general series by this purely formal procedure which later on was often performed unscrupulously with the help of the suggestive notation of Leibniz.

3. The binomial series. In an enclosure [10] with the letter to Collins of November 23, 1670, Gregory deals with the problem of finding the "number" of a given logarithm x; that is to say, if we denote the base by $1+d$, of finding $y=(1+d)^x$. For the sake of brevity, we again use modern notations without changing anything else. Gregory gives the solution as follows:

$$(1) \qquad (1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots,$$

which is of course the binomial series. The comparison of Gregory's formula and notation with the statement of the interpolation theorem in the principal part of the same letter [5] shows clearly that he found his result by applying the theorem to the function $f(x)=(1+d)^x$ using the known values at $x=0, 1, 2, \cdots$. Indeed, since the first difference of this function turns out to be

$$(2) \qquad \Delta f(x) = f(x+1) - f(x) = (1+d)^{x+1} - (1+d)^x = d\cdot f(x),$$

the values of its successive differences at $x=0$ become

$$f(0) = 1,\ \Delta f(0) = d,\ \Delta^2 f(0) = d^2,\ \Delta^3 f(0) = d^3, \cdots .$$

Thus, the interpolation formula **2**, (1) yields immediately the binomial series (1).

The correspondence of Gregory and Collins gives full evidence that this discovery of Gregory was entirely independent of Newton's investigations in the binomial theory. Gregory knew at this time only a single one of Newton's results, namely the series for the "zone of the circle," *i.e.* the series for the function $\int_0^x (R^2-x^2)^{1/2}dx$. Collins had communicated the mere statement of the latter to him seven months previously [**11**]. In fact, Newton had found this series by integrating term by term the expansion of the binomial $(R^2-x^2)^{1/2}$. Having Collins' communication, Gregory tried hard but without success to prove the result directly. Obviously, his discovery of the general binomial theorem was in no way influenced by this knowledge and he did not guess any connection. Afterwards, he recognized suddenly that Newton's series was a simple consequence of his own theorem and, in a letter of December 19 [**12**], complains much of "his own dullness," not to have noticed the fact before. Besides, Newton's binomial theorem did not become generally known before 1676, when, about ten years after he had found it, he communicated it to Oldenburg in the two famous letters [**13**] (June 6 and October 4).

It is interesting to compare the way in which Newton had discovered his theorem, as he describes it in the second of these letters, with Gregory's deduction. We mention only the most important points, simplifying the notation as before. Newton computes first the powers $(1+d)^n$ for the lowest integers $n=2, 3, 4, \cdots$, and discusses how to find directly the numerical coefficients of $d, d^2, d^3, \cdots$ in each of these expressions. He then makes the important remark that these coefficients in the expansion of $(1+d)^n$ can be generated by *multiplication* of the numbers $(n-0)/1, (n-1)/2, (n-2)/3, \cdots$, that is to say, that the coefficient of d^m in the expansion of $(1+d)^n$ is equal to

$$\frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot \cdots \cdot m}. \tag{3}$$

Of course, equivalent multiplicative relations for actually the same integers had been discovered a few years before by Pascal who defines them as elements of his "arithmetical triangle," without reference to the binomials.

From this statement Newton proceeds in an extremely audacious way. He got the idea from the procedure by which J. Wallis had developed his famous product formula for π by considering the successive integrals $\int_0^1 (1-x^2)^{n/2}dx$ for $n=0, 1, 2, \cdots$. (As a matter of fact, Newton starts in that letter with the consideration of these integrals instead of with the binomial itself.) He applies the same formula (3) also for the intermediate values $n=1/2, 3/2, 5/2, \cdots$ in order to obtain expressions for $(1+d)^n$ with these fractional values of the exponent, although he now has to write infinite series instead of finite sums. Further generalizations enable him to state the theorem for arbitrary values of the exponent.

To be sure, neither Gregory's nor Newton's deduction is an exact proof in the modern sense. In some respects, Gregory's way may seem to us more satisfactory: he deduces the result from a general theorem, the interpolation formula, and from a characteristic property of the function $(1+d)^x$, namely the difference equation (2). On the other hand, Newton makes this almost adventurous generalization of a finite algebraic identity, deduced for integral exponents only, into an infinite series for fractional exponents. Nevertheless, there is some internal connection between the two procedures. In his investigation, Newton considers the powers of a binomial as a function of the *exponent* as does Gregory, and not as a function of the second term d of the binomial. Thus, the procedures are not so different in their essence as they are in their execution. If one compares them with the usual modern proofs of the binomial theorem, one may remark that the latter are based on the consideration of $(1+d)^x$ as a function of d and that they use the successive *derivatives* with respect to d and the Taylor series instead of the successive *differences* with respect to x and the interpolation formula.

Newton realized the necessity of showing the way in which his consideration may be completed by a proper proof. As an example, he verifies by direct multiplication that the square of his series for $(1+d)^{1/2}$ is equal to $1+d$. Neither Gregory nor Newton tried to prove the convergence of the series. Such a proof was not, at this time, believed to be necessary; but certainly they had the feeling that these infinite sums determined definite numbers.

In this connection, it is interesting to find in a somewhat later letter of Gregory, dated April 9, 1672 [**14**] an early attempt to estimate the remainder of an infinite series by comparing it with the geometrical series. Here, he approximates the logarithmic series $x+x^3/3+x^5/5+x^7/7+\cdots$ by expressions such as

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{9x^7}{7\cdot 9-7\cdot 7x^2}$$

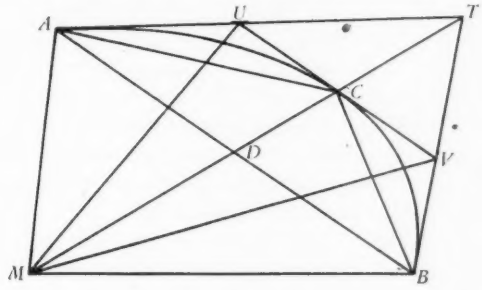
and emphasizes that the analogous expressions formed by using more terms of the original series will give a better approximation. Obviously, this estimate is obtained by comparison with the geometric series

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}\left(1+\frac{7x^2}{9}+\left(\frac{7x^2}{9}\right)^2+\cdots\right).$$

Thus, we see here the first step on the way which, more than a century later, led Cauchy to his convergence tests.

4. Gregory's "Vera Quadratura." Gregory's "*Vera Circuli et Hyperbolae Quadratura*" [**2**], a small volume, contains extremely interesting and original ideas which are, to be sure, a little remote from the mathematics of his time. Even if his mathematical technique was not always sufficient to get a complete solution of the problems he saw, even if he sometimes makes incomplete deductions and wrong conclusions, the investigations show an immense creative power.

He follows in some way the classical procedure of Archimedes, but reveals the algebraic content of the method. Besides, instead of calculating the perimeter of the circle as Archimedes did, he operates on areas. This enables him to deal simultaneously with the sectors of the circle, ellipse and hyperbola.



Let M be the center of a conic ACB, let AT and BT be the tangent lines at A and B, respectively, and let the straight line MT intersect AB at D and the conic at C. Gregory concludes first from fundamental properties of the conics the relations [15]:

$$AD = DB, \qquad MC^2 = MD \cdot MT. \tag{1}$$

Now he draws the tangent line at C which intersects AT at U and BT at V, and compares the following pairs of polygonal areas which are inscribed in or circumscribed about the sector $MACB$: on the one hand he compares the inscribed triangle $i_0 = MAB$ with the circumscribed quadrangle $I_0 = MATB$, on the other hand the inscribed polygon $i_1 = MACB$ with the circumscribed polygon $I_1 = MAUCVB$. The polygon i_1 is composed of two equal triangles MAC and MCB; the polygon I_1 of two equal quadrangles $MAUC$ and $MCVB$. Then, elementary properties of the conics, especially the relations (1), enable him to deduce easily two equations between these four areas as follows:

$$i_1 = \sqrt{i_0 I_0}, \qquad I_1 = \frac{2 i_1 I_0}{i_1 + I_0}.$$

Now, operating on the triangles MAC and MCB, and on the quadrangles $MAUC$ and $MCVB$ in the same way as he had operated on the triangles MAB and the quadrangle $MATB$, he gets four triangles of equal areas $i_2/4$, inscribed in the sector $MACB$, and four quadrangles of equal areas $I_2/4$ circumscribed about the same sector. Obviously, he obtains:

$$i_2 = \sqrt{i_1 I_1}, \qquad I_2 = \frac{2 i_2 I_1}{i_2 + I_1}.$$

Repeating the same operation n times, he constructs for each successive $n = 3, 4, \cdots$ an inscribed polygonal area i_n, composed of 2^n equal triangles,

and a circumscribed one I_n, composed of 2^n equal quadrangles. The successive areas are given by:

$$(2) \qquad i_{n+1} = \sqrt{i_n I_n}, \qquad I_{n+1} = \frac{2i_{n+1}I_n}{i_{n+1} + I_n} = \frac{2i_n I_n}{i_n + \sqrt{i_n I_n}} \qquad (n = 0, 1, 2, \cdots).$$

Geometrically it is obvious that the area S of the sector $MACB$ lies between each pair i_n, I_n, and that, if n increases indefinitely, these areas will approach S as closely as one desires, one sequence increasing from below, the other decreasing from above. But Gregory is not satisfied with this visual evidence. He recognizes in the successive construction of the i_n, I_n a new arithmetic operation which yields the value S, and therefore he feels a necessity to *prove* what we call the convergence of the limiting processes

$$(3) \qquad \lim_{n\to\infty} i_n = \lim_{n\to\infty} I_n = S.$$

In fact, with that high degree of exactness which we find in the classical Greek mathematics, he first shows that

$$|I_{n+1} - i_{n+1}| < \tfrac{1}{2}|I_n - i_n|$$

and then concludes that $|I_n - i_n|$ becomes smaller than any given number if n is sufficiently large.

To realize the mathematical importance of Gregory's method we may state that, for the circle and ellipse where $I_0 > i_0$, the area S can be expressed as follows:

$$(4) \qquad S = I_0\sqrt{\frac{i_0}{I_0 - i_0}}\ \text{arc tan}\ \sqrt{\frac{I_0 - i_0}{i_0}}.$$

For the circle, the first factor is simply $\frac{1}{2}MA^2$, the second the angle $\theta = BMA$. For the hyperbola where $I_0 < i_0$, we have only to interchange I_0 and i_0 and to replace the arc tangent function by the inverse of the hyperbolic tangent function. If we use imaginary numbers, we recognize that we have the same analytic function, since $\tanh ix = i \tan x$. But Gregory has discovered, without applying imaginary numbers, that the same analytical process—the approximation by the formulas (2), (3)—yields the area of the hyperbola as well as the area of the ellipse. In other words, he has found, for the first time in history, the analytical connection between the quadrature of sectors of the ellipse (or of the circle) and the quadrature of sectors of the hyperbola.

The history of these quadratures is interesting. We may assume that astronomical practice originally suggested the introduction of the arc of a circle as independent variable and the coordinates of the point on the circumference as dependent variables, that is to say, the introduction of the circular functions sine, cosine, and so on. This development may be connected with the fact that Archimedes investigated primarily the rectification of the circle instead of the quadrature. But the rectification of the general conics is an entirely different

and much more difficult problem. In considering the *area* of the circular sectors Gregory was able to find one single analytical process for the quadrature of all conics.

Now, it has been known since the middle of the 17th century that the quadrature of the hyperbola is connected with the logarithmic function. Therefore, it was obvious to Gregory himself that he had found *one* analytical process for getting from algebraic expressions to logarithmic functions as well as to inverses of the circular functions.

This discovery is generally ascribed to Euler who, some seventy years later, arrived at the connection between the exponential function and the circular functions by using formal operations in the field of complex numbers. It is doubtful whether Euler considered hyperbolic functions as analogous to circular functions and whether he used, in this respect, the analytical analogy between the processes of quadrature of circular and hyperbolic sectors.

The comparison of Euler's and Gregory's achievements may enhance our admiration for Gregory's genius. Indeed, it is not easy to connect in the field of real numbers the two integrals

$$\int \sqrt{1-x^2}\,dx \quad \text{and} \quad \int \sqrt{1+x^2}\,dx, \quad \text{or} \quad \int \frac{dx}{1+x^2} \quad \text{and} \quad \int \frac{dx}{1-x^2}.$$

As we have seen, this was achieved by Gregory.

In his "*appendicula ad veram circuli et hyperbolae quadraturam*" of 1668 [16] Gregory gives an array of linear combinations of the first i_n and I_n with definite numerical coefficients which yield much better approximations to the area S than do i_n and I_n themselves. Gregory was extremely offended that Huygens did not acknowledge his work to be an essential improvement over his older methods. Therefore he tried to make obvious the strength of the new theory by stating numerous new and surprising results without revealing how he had found them. Turnbull [17] has verified that, for the circle, one gets exactly Gregory's approximations if one first expresses i_n and I_n in terms of trigonometric functions of the angle θ, then expands these expressions in power series in θ, and finally forms such linear combinations of them which begin with the term θ and contain afterwards as many vanishing coefficients as possible. Analogous considerations are valid for the hyperbola. If Gregory operated in this manner he must have known the first terms of the power series for trigonometric and hyperbolic functions as early as 1668. Indeed, it is possible that he got this knowledge without using differentiation, but the published material does not seem to contain anything to corroborate this.

There are two other points in Gregory's speculations which particularly reveal the range of his mathematical ideas with respect to the actual later development of our science. First, the recurrent construction of the areas i_n, I_n is with him only *one* example of a very general, new analytic process which he coordinates as the "sixth" operation along with the five traditional operations

(addition, subtraction, multiplication, division, and extraction of roots). In the introduction, he proudly states "ut haec nostra inventio addat arithmeticae aliam operationem et geometriae aliam rationis speciem, ante incognitam orbi geometrico." This operation is, as a matter of fact, our modern limiting process. Clearly, his idea is, if we formulate it in modern language without changing the notions, to investigate two sequences of quantities $a_1 a_2, \cdots$ and $b_1, b_2, \cdots$, defined by the recurrent equations

$$a_{n+1} = \phi(a_n, b_n), \qquad b_{n+1} = \chi(a_n, b_n) \qquad (n = 1, 2, 3, \cdots). \tag{5}$$

He uses the word "convergent" for these sequences, very probably for the first time in history, if for each n

$$0 < b_{n+1} - a_{n+1} < b_n - a_n.$$

Of course, this definition does not conform completely to our precise notion of convergence; but in applying his notion he proves in most cases the correct and sufficient inequality

$$0 < b_{n+1} - a_{n+1} < \rho(b_n - a_n)$$

where $\rho < 1$ is independent of n. (In his original problem, he has, as seen previously, $\rho = \frac{1}{2}$.) Then he concludes that the "last convergent terms" of the sequences a_n and b_n are equal, and he calls them *terminatio* of the sequences. In his original problem this terminatio is the area S.

From his further examples we may mention the following ones:

$$a_{n+1} = a_n + \alpha(b_n - a_n), \qquad b_{n+1} = b_n - \beta(b_n - a_n) \tag{6}$$

and

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}. \tag{7}$$

Here he succeeds in finding the terminatio by an ingenious and simple idea: he determines an invariant expression $F(a_n, b_n)$ such that

$$F(a_{n+1}, b_{n+1}) = F(a_n, b_n); \tag{8}$$

then, the terminatio t will satisfy the equation

$$F(a_1, b_1) = F(t, t), \tag{9}$$

which gives the value t in terms of a_1 and b_1. For the examples (6), (7) he can state immediately the invariant expressions $F(a_n, b_n) = \beta a_n + \alpha b_n$ and $F(a_n, b_n) = a_n \cdot b_n$, respectively, and he finds as the termatio, using (9):

$$t = \frac{\beta a_1 + \alpha b_1}{\beta + \alpha} \quad \text{and} \quad t = \sqrt{a_1 b_1},$$

respectively.

One may remark that Gregory investigated in (2) and (7) different combinations of arithmetical, geometrical and harmonical means. One could imagine that he tried to treat other combinations of these means, but that he could not find out an algebraic expression or a geometric interpretation. In the following century the relation between the arithmetical-geometrical mean and the elliptic integrals was discovered by Lagrange, Legendre and Gauss. We know especially that Gauss studied these means in his early youth before he had any knowledge of the calculus, and that these means, later on, showed him the way to the elliptic integrals [18]. We know moreover that Pfaff, the teacher of Gauss, investigated sequences closely related to Gregory's sequence (2) [19]. Thus, we could guess that we have here an influence of Gregory's work on one of the most important theories of modern analysis, but we have no definite evidence of such connections.

The second point may be still more momentous. Gregory attempts to prove that the terminatio S of the polygons i_n, I_n cannot be expressed by using the traditional five "elementary" operations on i_0 and I_0. In the preface he puts particular emphasis on this phenomenon. From his exposition we may suppose that he first had tried to "square the circle," *i.e.* to find such an "elementary" expression for S. But he was critical enough to recognize that the difficulties in this search could not be overcome. And realizing that the task of algebra and analysis consists as well in solving a problem as in proving, if necessary, the "impossibility" of a certain solution, he dared to try such a proof, although he did not find any pattern for doing it. He emphasizes that since Euclid's classification of the usual irrationalities in his tenth book, nothing of this kind has even been attempted. Of course, Leonardo Pisano had shown [20] about 1200 A.D., that a certain cubic equation cannot be solved by Euclid's irrationalities. However, Gregory could not have had any knowledge of this investigation since it was not published before the nineteenth century. It is a testimony to Gregory's surprising intuition that he mentions further as problems impossible in the same sense just these two: to solve the general algebraic equation and to get an nth root by solving quadratic equations.

To be sure, Gregory does not prove that it is impossible to square the circle, although this is in his mind. He approaches only a much easier problem: to prove that the area of an arbitrary circular sector S cannot be expressed in terms of the areas i_0 and I_0 by the five elementary operations—or, in modern language, that the arc tangent function as given by (4) and defined by the limiting process (2), (3), is not a combination of such algebraic functions. The foundation of his proof is the remark that two sequences (2) yield the same terminatio S whether we begin the process with i_0, I_0 or with i_1, I_1; therefore S depends upon i_0 and I_0 in the same way as upon i_1 and I_1. To put it in modern language, the function satisfies the algebraic functional equation:

$$S(i_0, I_0) = S(i_1, I_1) = S\left(\sqrt{i_0 I_0},\ \frac{2 i_0 I_0}{i_0 + \sqrt{i_0 I_0}}\right), \tag{10}$$

i.e. $S(i_0, I_0)$ can be transformed algebraically into itself. He tries to prove that the identity (10) is impossible for any function formed only by the five elementary operations. First he removes the irrationality, introducing two suitable new variables u, v by the equations

$$i_0 = u^2(u + v), \qquad I_0 = v^2(u + v).$$

Then (2) shows that

$$i_1 = uv(u + v), \qquad I_1 = 2uv^2,$$

and the identity (10) becomes

$$S(u^2(u + v), v^2(u + v)) = S(uv(u + v), 2uv^2). \tag{11}$$

Now he states two properties of this identity from which he is going to deduce its impossibility for functions S of the above described algebraic type: 1) The arguments of S on the left side contain u up to the third power, while those on the right side contain u only up to the second power. 2) On the left side, both arguments are binomial, while on the right side the second one is only monomial.

Of course, Gregory is able to prove correctly by this procedure that the identity (11) cannot be satisfied by a rational integral function S of its two arguments, and even, with slightly more difficulty, that it cannot be satisfied by any rational function. However, we do not believe that the facts he offers are sufficient to furnish the proof that S is not an irrational function built up in using extraction of roots. Indeed, the algebraic factor $I_0\sqrt{i_0}/\sqrt{I_0 - i_0}$ of (4) satisfies, itself, an identity which differs from (10) only by a factor 2 in the left member, and Gregory's considerations could be applied equally well to the modified identity. The point is that the identity (10), used as basis for his proof implies an intrinsic difficulty: it is equivalent to the algebraic relation between $\tan\theta$ and $\tan 2\theta$ and, moreover, Gregory thinks of it only as valid in the restricted interval $0 < \theta < \frac{1}{2}\pi$.

Today, we would conclude the transcendental character of $\tan\theta$ (and, simultaneously, of the inverse function arc tangent) immediately from the periodicity of that function ($\tan\theta = \tan(\theta + \pi)$). Although such a conclusion seems to us extremely simple, it may have been difficult and remote at Gregory's time.

A modern mathematician will highly admire Gregory's daring attempt of a "proof of impossibility" even if Gregory could not attain his aim. He will consider it a first step into a new group of mathematical questions which became extremely important in the 19th century. However, the contemporary echoes of Gregory's undertaking were in no way favorable. First of all, Huygens criticized [21] the "*Vera Quadratura*" in an extremely unfavorable manner. Gregory had sent him one of the first copies. He expected his discoveries to be fully appreciated by this great mathematician who himself had done very important work on the problem of the quadrature of conics and the circle. But, unfortunately, Huygens was apparently angry that those earlier investigations were not mentioned. Thus, he put more emphasis on some claims of priority and on

some objections against Gregory's deductions than on the importance of Gregory's new ideas and results. There is no need to report here on the unpleasant discussion which arose from this criticism [22]. We mention only the single point of importance where Huygens showed a profounder insight. He says: even if the area of an arbitrary circular sector cannot be expressed algebraically in terms of the areas i_0, I_0, one can still imagine such an expression to be possible for particular sectors, for example, for the whole circle itself. Gregory, obviously, had overlooked this possibility in his original publication. In his answer he tried to deduce the result for the "particular case" from that for the arbitrary sector. These endeavors could not but fail; it took more than two centuries before mathematics had developed the necessary means to prove the transcendency of π.

5. Conclusion. Surveying the importance of all these discoveries and ideas of Gregory, and realizing that the total range of his scientific work is by no means covered by our report, one may wonder why this great man did not exert more influence on the actual development of mathematics. The reason can be found in some unfortunate, almost tragical facts in Gregory's life which hampered his activity as well as the effectiveness of his work. After some short sojourns in London (1663 and 1668), and several years of inspiring studies in Italy (1664–1668), mostly in Padua, he was appointed Professor of Mathematics at the Scottish University of St. Andrews. At this old school, still living entirely in medieval traditions, the young scholar was rather isolated. There he was the only one who knew of the new development of mathematics. He himself abounded with new ideas, but there was no possibility to discuss or to teach them. Moreover, hardly any literature was available. Only through his correspondence with Collins whom he had met in London and who had become his close friend, could he learn what the great mathematicians in England and abroad were planning and completing.

Thus, his ideas could not find the response they deserved and he himself did not develop them as far as it might have been possible in closer contact with mathematicians of equal rank. Still worse consequences may have been involved in the lack of appreciation of his first important publication, the *Vera Quadratura*, and especially in the unkind and unjust criticism of Huygens which we have mentioned above.

Apparently, these experiences impressed the proud young Scotchman so deeply that he abandoned entirely the trend of ideas he had started so successfully. We can imagine that otherwise he might have applied his "convergent" pairs of sequences, as defined by recurrence formulas, to various problems and that he might have brought this important process to greater prominence in the early analysis. In fact, he afterwards used the infinite series, probably influenced by the reports he got, scantily, on Newton's work. Yet, also here, fate did not favor him. For he was not given time and opportunity to complete and publish his investigations; and his great merits were darkened by Newton's glory who, meanwhile, could finish his work.

Besides, Gregory had inaugurated research on differential and integral calculus without knowing what his eminent competitors were doing simultaneously in this field. He was even the first to publish, as early as 1668, a proof [23] of the "fundamental theorem," that the two characteristic problems of the calculus, namely, to determine the slopes and to determine the areas, are inverse to one another. Also here he met misfortune; immediately afterwards there appeared Barrow's great work "lectiones geometricae," which went much farther and won all fame. A few years later, Newton's and Leibniz's momentous results on the calculus became known and made obsolete the work of all their predecessors.

Gregory did not live to see this development. He had eventually taken over a professorship at the University of Edinburgh, which granted him better working opportunities. But only one year later, in the fall of 1675, he suddenly fell ill and died in his thirty-seventh year. Most of his discoveries and ideas were buried in his letters and notes or lost through his death.

References

1. Published for the Royal Society of Edinburgh, London, 1939.

2. Pataviae, 1667; reprinted as appendix to J. Gregory's Geometria pars Universalis, Venetiae, 1668, and again in Chr. Huygens, Opera varia, vol. II, Lugduni Batavorum, 1724, pp. 407–462. Our report in no. 4 is based on our essay in the Memorial [1], pp. 468–478.

3. N. Mercator, Logarithmotechnica, Londini, 1668.

4. Published with a comprehensive commentary of H. W. Turnbull in the Memorial [1], pp. 350–359.

5. Memorial [1], pp. 118–122, especially p. 119 f.; *cf.*, Turnbull's commentary, *ibid.*, p. 124. With regard to the earlier publications of that letter, one may compare *ibid.*, pp. 25 and 29.

6. It is mentioned first, but not formulated, in Newton's letter to Oldenburg of October 24, 1676 (Newton, Opuscula Mathematica I, Lausannae et Genevae, 1744, pp. 328–357; see particularly p. 340) and completely stated in his Philosophiae Naturalis Principia, 1687, book III, lemma V, and in his Methodus Differentialis, 1711 (Opuscula [6], p. 271 ff.)—at both places for equidistant and non-equidistant ordinates.

7. See Memorial [1], p. 58, and Turnbull's note, p. 59.

8. In the letter quoted in [6], p. 341.

9. Methodus Incrementorum, Londini, 1715.

10. See Memorial [1], p. 131 f., and Turnbull's commentary, p. 132 f.

11. Letter of March 24, 1670, Memorial [1], p. 88; *cf.*, Gregory's answer, *ibid.*, p. 92.

12. Memorial [1], p. 148; *cf.*, Turnbull's commentary, p. 150 f.

13. Newton, Opuscula I [6], pp. 307–322, especially p. 307 f., and pp. 328–357, especially pp. 329 ff.

14. Memorial [1], p. 230.

15. In our essay in the Memorial [2], p. 469, the second of these formulas is misprinted. We may correct here some other minor misprints in that essay: p. 469, last line, read I_2 instead of $\sqrt{I_2 \cdot i_2}$; p. 471, last formula, read $b_{n+1}=\chi(a_n, b_n)$ instead of $b_{n+1}=\phi(a_n, b_n)$; p. 478, note 7, read tanh instead of tan on one side of the formula.

16. First part of Gregory's Exercitationes Geometricae, Londini, 1668.

17. See Memorial [1], p. 461 ff.

18. *Cf.*, L. Schlesinger, Gauss' Fragmente zur Theorie des arithmetisch-geometrischen Mittels, Nachrichten der Goettinger Gesellschaft der Wissenschaften, 1912, and his essay in Gauss, Werke, vol. X, part 2, Berlin, 1933, Abhandlung 2.

19. *Cf.*, Pfaff's letters to Gauss in Gauss, Werke, vol. X, part 1, Leipzig, 1917, p. 234 ff., and H. Geppert, Mathematische Annalen, vol. 108, 1933, p. 205 ff.

20. Published first by B. Boncompagni, tre scritti inediti di Leonardo Pisano, Firenze, 1854. The proof to which we refer is reviewed comprehensivly by F. Woepke, Journal de Mathématiques pures et appliquées, vol. 19, 1854, p. 401 ff.

21. First in a review in the Journal des Sçavans, Paris, July, 1668; *cf.*, the references in [22].

22. One may compare, also for further literature, our essay [2] and the comprehensive report of E. J. Dijksterhuis, Memorial [1], pp. 478–486. The most important parts of the discussion are reprinted in Chr. Huygens, Opera varia II [2], pp. 463 ff., and again in the Oeuvres complètes de Christiaan Huygens, vol. VI, La Haye, 1895, pp. 228 ff.

23. Contained in the Geometriae pars universalis, Venetiae, 1668. *Cf.*, the essay of A. Prag on this work in the Memorial [1], pp. 487 ff.

THE CLOSURE OF SYSTEMS OF ORTHOGONAL FUNCTIONS

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1. Introduction. A system of functions $\phi_n(x)$, $n=0, 1, 2, \cdots$ is said to be *orthonormal* on the finite interval (a, b) provided that

$$\int_a^b \phi_n(x)\phi_m(x)dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$

With any integrable function $g(x)$ there is an associated generalized Fourier series

$$g(x) \sim \sum_{n=0}^{\infty} a_n\phi_n(x), \qquad a_n = \int_a^b g(x)\phi_n(x)dx. \tag{1}$$

An orthonormal system is said to be *closed in the class H* of functions if the series (1) associated with an arbitrary function $g(x)$ in H converges in the mean to $g(x)$; that is, if

$$\lim_{n\to\infty} \int_a^b \{g(x) - s_n(x)\}^2 dx = 0 \tag{2}$$

where $s_n(x)$ denotes the sum of the first n terms of (1).

The importance of the concept of closure in teaching courses involving orthogonal series is quite generally recognized. Various conditions for the validity of the property are known. Unfortunately, the application of those conditions to specific orthogonal systems is, even in the simplest cases, somewhat abstruse for presentation to a class composed of, say, seniors and first year graduate students in physics and engineering. In the second section of this paper a new criterion for closure is given which can be applied directly to verify the property for a number of classical orthogonal systems. In the third section the application of the condition is indicated for the system of Legendre polynomials and the system of trigonometric functions. The entire procedure may be shown to students having no more preparation than a course in Advanced Calculus.

The idea used in the derivation of the criterion mentioned is not essentially new. A variation of it was applied to the trigonometric system by Tamarkin [4]* and to certain systems of orthogonal polynomials by Sansone [3].

Integration is used throughout the paper in the Riemann sense and all integrals employed will be assumed to exist without further mention. It will be evident to the reader who is interested in generality that the results derived here may be applied to the question of the closure of orthonormal systems in the class L^2 of functions whose squares are Lebesque integrable. Generalizations along other lines are indicated in the last section of the paper.

2. Criteria for closure. In what follows the *class G* will be used as a generic notation for the class of all functions which are bounded and integrable on the interval (a, b).

The point of departure for the proof in this section is a result due to Vitali [5]:

THEOREM 1. *A necessary and sufficient condition that the orthonormal system* $\{\phi_n(x)\}$ *of functions in the class G be closed in that class*† *is that, for every t in the closed interval* (a, b),

$$t - a = \sum_{n=0}^{\infty} \left(\int_a^t \phi_n(x)dx \right)^2. \tag{3}$$

This result is an easy consequence of the following sequence of well known lemmas:‡

LEMMA 1. *A necessary and sufficient condition that the system* $\{\phi_n(x)\}$ *be closed in any subclass H of G is that the Parseval formula*

$$\int_a^b g^2(x)dx = \sum_{n=0}^{\infty} a_n^2, \qquad a_n = \int_a^b g(x)\phi_n(x)dx, \tag{4}$$

be valid for every function $g(x)$ *in the class H.*

LEMMA 2. *If the system* $\{\phi_n(x)\}$ *is closed in any class* H_0 *of functions, then it is closed in the class* H_1 *composed of all finite linear combinations (with constant coefficients) of functions extracted from* H_0.

LEMMA 3. *If* $g(x)$ *is chosen arbitrarily from the class G and if* ϵ *is an arbitrary positive number, there exists a step function* $s(x)$ [*a function constant in value over each subinterval of a finite subdivision of* (a, b)], *such that*

$$\int_a^b \{g(x) - s(x)\}^2 dx < \epsilon.$$

LEMMA 4. *Let* (a) $g(x)$ *be an arbitrary function in the class G,* (b) $s_n(x)$ *denote*

* The numbers in brackets refer to the bibliography.

† The theorem remains true if G is replaced by L^2.

‡ For an alternative proof see Tamarkin [4].

the sum of the first n terms of the series (1) *for* $g(x)$, *and* (c) $L_n(x)$ *be any linear combination whatever of the first n functions of the set* $\{\phi_n(x)\}$. *Then*

$$\int_a^b \{g(x) - s_n(x)\}^2 dx \leqq \int_a^b \{g(x) - L_n(x)\}^2 dx.$$

The proofs of these theorems are well known. For Lemmas 1, 2, and 4 see Kaczmarz and Steinhaus [1]. Lemma 3 is an immediate consequence of the definition of the Riemann integral.

Now since (3) is the specialization of (4) to the function

$$g_t(x) = \begin{cases} 1 & \text{on} \quad a \leqq x \leqq t, \\ 0 & \text{on} \quad t < x \leqq b, \end{cases} \tag{5}$$

by Lemma 1, if (3) holds for every t, the system $\{\phi_n(x)\}$ is closed in the class H_0 of all functions given by (5). Any step function is a linear combination of the functions $g_t(x)$; thus, by Lemma 2, the system is closed in the class of all step functions. By Lemma 3, given any $g(x)$ in G and any $\epsilon > 0$, one can find a step function $s(x)$ such that

$$\left[\int_a^b \{g(x) - s(x)\}^2 dx\right]^{1/2} < \tfrac{1}{2}\epsilon.$$

Let $s_n(x)$ denote the sum of the first n terms of the series (1) for the step function $s(x)$. There is a number N such

$$\left[\int_a^b \{s(x) - s_n(x)\}^2 dx\right]^{1/2} < \tfrac{1}{2}\epsilon.$$

if $n \geqq N$. It follows by Minkowski's inequality* that

$$\left[\int_a^b \{g(x) - s_n(x)\}^2 dx\right]^{1/2} < \epsilon.$$

Now let $\sigma_n(x)$ denote the sum of the first n terms of the series (1) for the function $g(x)$. By Lemma 4 one has

$$\int_a^b \{g(x) - \sigma_n(x)\}^2 dx \leqq \int_a^b \{g(x) - s_n(x)\}^2 dx.$$

This proves the sufficiency clause. The necessity is evident.

The sum of the first n terms of the series on the right in (3) can be written as

$$\sum_{r=0}^{n} \int_a^t \left\{\frac{d}{dt}\left(\int_a^t \phi_r(x)dx\right)^2\right\} dt = \int_a^t \left\{2\sum_{r=0}^{n} \phi_r(t) \int_a^t \phi_r(x)dx\right\} dt.$$

Thus, if one defines the function

* The Minkowski inequality with exponent 2 is an easy consequence of Schwarz inequality. Kaczmarz and Steinhaus [1].

$$\sigma_n(t) = \sum_{r=0}^{n} \phi_r(t) \int_a^t \phi_r(x)dx, \tag{6}$$

the formula (3) becomes equivalent to the condition

$$\lim_{n\to\infty} \left| \int_a^t \{1 - 2\sigma_n(t)\}dt \right| = 0 \text{ for all } t \text{ in } (a, b).$$

Now, for every t in (a, b) one has

$$\begin{aligned} \left| \int_a^t \{1 - 2\sigma_n(t)\}dt \right| &\leqq \int_a^t |\, 1 - 2\sigma_n(t)\,|\, dt \\ &\leqq \int_a^b |\, 1 - 2\sigma_n(t)\,|\, dt \\ &\leqq \left\{ \int_a^b dt \int_a^b \{1 - 2\sigma_n(t)\}^2 dt \right\}^{1/2} \end{aligned}$$

where Schwarz' inequality has been used in the last step. This establishes

THEOREM 2. *A sufficient condition that the orthonormal system* $\{\phi_n(x)\}$ *of functions in the class G be closed in that class is that*

$$\lim_{n\to\infty} \int_a^b \{1 - 2\sigma_n(t)\}^2 dt = 0, \tag{7}$$

where the function $\sigma_n(t)$ *is defined by* (6); *that is, that the series*

$$\sum_{n=0}^{\infty} \phi_n(t) \int_a^t \phi_n(x)dx$$

converge in the mean to the value $\frac{1}{2}$.

3. Applications. *The Legendre polynomials.* The Legendre polynomials satisfy the relations

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2/(2n+1) & \text{if } n = m. \end{cases} \tag{8}$$

In this case the function $\sigma_n(t)$ is given by

$$\sigma_n(t) = \sum_{r=0}^{n} \{(2r+1)/2\} P_r(t) \int_{-1}^{t} P_r(x)dx.$$

Using the properties

$$(2r+1)P_r(x) = \frac{d}{dx}\{P_{r+1}(x) - P_{r-1}(x)\},$$

$$P_r(1) = 1, \qquad P_r(-1) = (-1)^r,$$

it is found that

$$\sigma_n(t) = \frac{1}{2}\left\{P_0(t)P_1(t) + \sum_{r=1}^{n} P_r(t)\left[P_{r+1}(t) - P_{r-1}(t)\right]\right\}$$

$$= \frac{1}{2}\left\{1 + P_n(t)P_{n+1}(t)\right\}.$$

The integral in (7) becomes (by (8))

$$\int_{-1}^{1} P_n^2(t)P_{n+1}^2(t)dt \leqq \int_{-1}^{1} P_{n+1}^2(t)dt = 2/(2n+3)$$

where the simple bound $|P_n(t)| \leqq 1$ has been used. The condition (7) of theorem 2 is evidently satisfied.

The trigonometric functions. Perhaps the most important orthogonal system is composed of the functions

$$\tfrac{1}{2}, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots.$$

The standard notation

$$g(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin nx dx,$$

differs from (1). To apply theorem 2 here note that the function $\sigma_n(t)$ given in (6) is merely the sum of the first n terms, evaluated at $x=t$, of the series (1) for the function $g_t(x)$ defined in (5). Using $(0, 2\pi)$ as the interval of orthogonality, it is found that

$$\sigma_n(t) = \frac{t}{2\pi} + \frac{1}{\pi}\sum_{r=1}^{n}\left\{\cos rt\int_0^t \cos rx\,dx + \sin rt\int_0^t \sin rx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{2}t + \sum_{r=1}^{n}\frac{\sin rt}{r}\right\}.$$

Squaring the quantity $\{1-2\sigma_n(t)\}$ and integrating over $(0, 2\pi)$, the condition (7) is found to be

$$\lim_{n\to\infty}\left\{\frac{\pi}{6} - \frac{1}{\pi}\sum_{r=1}^{n}\frac{1}{r^2}\right\} = 0.$$

Since the formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \tag{9}$$

is well known, the condition in question is fulfilled.

There are both simple proofs and elementary proofs for the equation (9) but the author is not aware of one that combines these virtues and is at the same time independent of the theory of Fourier series. Since the principal thesis of this paper is simplicity, an elementary proof of (9) will be given here. It has been borrowed from an unpublished manuscript by Marjorie Leffler [2]. The fact that the following Lemma can be proved in the simple manner shown below does not seem to be widely known.

LEMMA 5. *The Fourier series*

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n} = \begin{cases} \frac{1}{2}(\pi - t) & \text{if } 0 < t < 2\pi, \\ 0 & \text{if } t = 0 \text{ or } 2\pi, \end{cases} \tag{10}$$

and the convergence of the series is uniform on the interval $\epsilon \leqq t \leqq 2\pi - \epsilon$ *for arbitrary positive* ϵ.

Proof. Integration of the trigonometric identity

$$\frac{1}{2} + \sum_{r=1}^{n} \cos rx = \frac{\sin (n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}$$

over the interval (t, π) yields

$$\frac{1}{2}(\pi - t) - \sum_{r=1}^{n} \frac{\sin rt}{r} = \frac{1}{2}\int_t^{\pi} \frac{\sin (n + \frac{1}{2})x\,dx}{\sin \frac{1}{2}x}. \tag{11}$$

If $\epsilon > 0$ is arbitrary and $\epsilon \leqq t \leqq 2\pi - \epsilon$, the integral on the right in (11) may be integrated by parts to obtain

$$\begin{aligned} \left| \int_t^{\pi} \frac{\sin (n + \frac{1}{2})x\,dx}{\sin \frac{1}{2}x} \right| &= \left| \frac{2 \cos (n + \frac{1}{2})t}{(2n + 1) \sin \frac{1}{2}t} + \frac{1}{2n + 1} \int_t^{\pi} \frac{\cos \frac{1}{2}x \cos (n + \frac{1}{2})x\,dx}{\sin^2 \frac{1}{2}x} \right| \\ &\leqq \frac{2}{(2n + 1)\left| \sin \frac{1}{2}t \right|} + \frac{1}{2n + 1} \left| \int_t^{\pi} \frac{dx}{\sin^2 \frac{1}{2}x} \right| \\ &\leqq \frac{2}{(2n + 1) \sin \frac{1}{2}\epsilon} + \frac{\pi}{(2n + 1) \sin^2 \frac{1}{2}\epsilon}. \end{aligned}$$

The remainder of the proof is now obvious.

By termwise integration of (10) over (t, π) it is found that

$$\frac{1}{4}t^2 - \frac{\pi}{2}t + \frac{1}{4}\pi^2 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nt \tag{12}$$

is valid on the interval $\epsilon \leqq t \leqq 2\pi - \epsilon$ for arbitrary positive ϵ. Since ϵ is arbitrary and both sides of (12) are continuous functions on the closed interval $0 \leqq t \leqq 2\pi$, the equation (12) must persist over the entire interval. Now let

$$\sigma = \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sigma_e = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}, \qquad \sigma_0 = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Clearly

$$\sigma_e = \tfrac{1}{4}\sigma, \quad \sigma_0 = \tfrac{3}{4}\sigma, \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = \sigma_e - \sigma_0 = -\tfrac{1}{2}\sigma.$$

Using the last formula in (12) with $t=0$, the equation (9) is immediate.

Incidentally, the final form of (12) verifies condition (3) of Theorem 1 directly.

4. Generalizations. Theorem 2 can be extended to apply to systems of functions $\{\phi_n(x)\}$ which are orthogonal on (a, b) with respect to a non-negative weight function $w(x)$. In this instance the Parseval formula (4) is replaced by

$$\int_a^b w(x)g^2(x)dx = \sum_{n=0}^{\infty} a_n^2, \qquad a_n = \int_a^b w(x)g(x)\phi_n(x)dx.$$

An argument entirely analogous to that used in deriving (7) furnishes the closure condition

$$\lim_{n\to\infty} \int_a^b w(x)\{1 - 2\sigma_n(t)\}^2 dt = 0$$

where

$$\sigma_n(t) = \sum_{r=0}^{n} \phi_r(t) \int_a^t w(x)\phi_r(x)dx.$$

By placing proper restrictions upon the functions involved, the interval (a, b) may be allowed to be infinite.

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GRAPHICAL SOLUTIONS OF CUBIC, QUARTIC, AND QUINTIC

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1. Statement of the problem. It is proposed in this paper to develop a method for the graphical solution of equations of the 3rd, 4th, and 5th degrees with real coefficients. The roots are found as abscissas of the points of intersection of two curves whose equations are easily obtained from the equation to be solved. We will begin with the solution of the quartic in x. The cubic may be solved as a quartic after multiplying through by x. The solution of the quintic will be taken up after the quartic has been solved. It will be seen that the method may be applied to the solution of equations of higher degree provided they have no more than four complex roots.

2. Basis of solution. Let the roots of the quartic be represented by $a \pm bi$ and $c \pm di$. The equation may then be written

$$\text{(A)} \qquad \begin{aligned} &x^4 - 2(a+c)x^3 + (a^2+b^2+c^2+d^2+4ac)x^2 \\ &\quad - 2[a(c^2+d^2) + c(a^2+b^2)]x + (a^2+b^2)(c^2+d^2) = 0. \end{aligned}$$

This is identically

$$\begin{aligned} &[x^2 - (a+c)x + \tfrac{1}{2}(d^2+b^2+2ac)]^2 \\ &\quad - [(d^2-b^2)(a-c)x + \tfrac{1}{4}(d^2-b^2)^2 + (b^2c - ad^2)(a-c)] = 0. \end{aligned}$$

Transposing the second bracket and extracting the square root of both members we see that equation (A) is satisfied by the abscissas of the points of intersection of the two parabolas

$$\text{(1)} \qquad y = x^2 - (a+c)x + \tfrac{1}{2}(d^2+b^2+2ac) \quad \text{and}$$

$$\text{(2)} \qquad y^2 = (d^2-b^2)(a-c)x + \tfrac{1}{4}(d^2-b^2)^2 + (b^2c - ad^2)(a-c).$$

The parabolas (1) and (2) will be referred to as auxiliary curves. To obtain the right hand member of (1) extract the square root of the left hand member of equation (A) stopping at the third term. The square of the right hand member of (1) minus the left hand member of (A) gives the right hand member of (2).

3. Real roots. If the auxiliary curves intersect in four points the four roots are read as abscissas of the points of intersection. If they intersect in only two points the two real roots are read from the diagram and the complex roots are easily obtained.

4. Complex roots. If the auxiliary curves do not intersect the roots are complex. These roots will be obtained by employing a new set of auxiliary curves. These curves will give us the value $a-c$. Rewrite equations (1) and (2) in the form

$$\text{(3)} \qquad y = x^2 + Ax + B,$$

$$\text{(4)} \qquad y^2 = Cx + D.$$

By equating the coefficients in (3) and (4) to those in (1) and (2) and eliminating b^2 and d^2 we obtain the sextic (5) in $a-c$.

$$(5) \qquad (a-c)^6+(4B-A^2)(a-c)^4+(4D-2AC)(a-c)^2-C^2=0 \quad \text{or}$$

$$(6) \qquad z^3+(4B-A^2)z^2+(4D-2AC)z-C^2=0$$

where

$$z=(a-c)^2, \qquad b^2=B-ac-\frac{C}{2(a-c)}, \qquad d^2=B-ac+\frac{C}{2(a-c)}.$$

It is easily shown that equation (6) has one and only one positive root. Multiplying (6) through by z we have the quartic equation

$$(7) \qquad z^4+(4B-A^2)z^3+(4D-2AC)z^2-C^2z=0$$

to solve for the value of $(a-c)^2$. The auxiliary curves obtained for the solution of (7) are

$$(8) \qquad y=z^2+(2B-\tfrac{1}{2}A^2)z+2D-AC-\tfrac{1}{2}(2B-\tfrac{1}{2}A^2)^2 \quad \text{and}$$

$$(9) \qquad y^2=\{(2B-\tfrac{1}{2}A^2)[4D-2AC-(2B-\tfrac{1}{2}A^2)^2]+C^2\}z+[2D-AC-\tfrac{1}{2}(2B-\tfrac{1}{2}A^2)^2]^2.$$

As we are concerned with only the positive root of (7) the diagram in any particular case will be quite simple. The introduced root zero is disregarded. It is well to observe that if $C=0$, $a-c$ is not necessarily zero. Either $a=c$ or $d^2=b^2$ will make $C=0$. If $C=0$ and D negative the value of $(a-c)^2$ is found from the equation $z^2+(4B-A^2)z+4D=0$.

Example. Solve $x^4-2x^3+3x^2-4x+5=0$.

The auxiliary curves are

$$(10) \qquad y=x^2-x+1, \qquad A=-1, \qquad C=2,$$

$$(11) \qquad y^2=2x-4, \qquad B=1, \qquad D=-4.$$

From (7) we have

$$(12) \qquad z^4+3z^3-12z^2-4z=0.$$

The roots of equation (12) are the abscissas of the points of intersection of the two auxiliary curves

$$y=z^2+\tfrac{3}{8}z-7\tfrac{1}{8} \quad \text{and}$$

$$y^2=-17\tfrac{3}{8}z+50\tfrac{49}{64}.$$

From the graphs of the two equations z is found to be 2.48. The complex roots are then found in the following way:

$$(a-c)^2=2.48 \qquad c=-0.29$$

$$a-c=1.57 \qquad ac=-0.37$$

$$a+c=1.00 \qquad b^2=0.74$$

$$a = 1.29 \qquad d^2 = 2.00$$

$$x = 1.29 \pm 0.86i \quad \text{or} \quad -0.29 \pm 1.42i.$$

The solution of the cubic is illustrated by the solution of equation (12) and needs no further explanation.

The solution of the quintic is best set forth by an

Example. Solve $x^5-4x^4+7x^3-10x^2+13x-10=0$. Introducing the factor x we obtain the equation

$$x^6 - 4x^5 + 7x^4 - 10x^3 + 13x^2 - 10x = 0.$$

By a process similar to the one used in obtaining equations (1) and (2) it becomes evident that the roots of the sextic are the abscissas of the points of intersection of the auxiliary curves

(13) $$y = x^3 - 2x^2 + \tfrac{3}{2}x - 2 \quad \text{and}$$

(14) $$y^2 = -\tfrac{11}{4}x^2 + 4x + 4.$$

From the graphs of these equations it is seen that (omitting the introduced value zero) there is only one point of intersection. The abscissa of this point is 2. Reducing the degree of the given quintic we obtain the quartic just solved.

It might be remarked that in the case of double roots the auxiliary curves are tangent to each other, and in the case of triple roots they cross with a common tangent. The graphs of the auxiliary curves can only lead us to suspect the presence of double or triple roots if they exist. Nearly equal roots cannot be separated graphically.

5. Equations of higher degree. The preceding naturally suggests the possibility of a graphical solution of equations of higher degree with real coefficients proved they have no more than four complex roots. Let the general equation be written

$$x^{2n} + A_1x^{2n-1} + A_2x^{2n-2} + A_3x^{2n-3} + A_4x^{2n-4} + \cdots + A_{2n-1}x + A_{2n} = 0.$$

If an equation given for solution is of an odd degree it must be made of an even degree by introducing the factor x. The equations of the auxiliary curves, obtained in a way similar to the way (1) and (2) we obtained, are

(15) $$y = x^n + B_1x^{n-1} + B_2x^{n-2} + B_3x^{n-3} + \cdots + B_{n-1}x + B_n \quad \text{and}$$

(16) $$y^2 = C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \cdots + C_{n-1}x + C_n.$$

To obtain the right hand member of (15) extract the square root of the left hand member of the general equation stopping with the $(n+1)$th term, then proceed as before.

Example. Let it be required to solve the equation

$$x^8 - 10x^6 + 33x^4 - 40x^2 - \tfrac{2}{3}x + 15\tfrac{4}{9} = 0.$$

The auxiliary curves found from the given equation are

$$y = x^4 - 5x^2 + 4, \tag{17}$$

$$y^2 = \tfrac{8}{5}x + \tfrac{5}{8}. \tag{18}$$

By carefully graphing equations (17) and (18) on cross section paper, say twenty divisions to the unit, it will be seen that the abscissas of the eight points of intersection are -2.03, -1.97, -1.10, -0.90, 0.85, 1.15, 1.90, and 2.07. The last figures in the roots are somewhat in doubt. In the case of complex roots, not more than four, the solution of the quintic illustrates the method of procedure. Only rough approximations to the roots can be obtained graphically.

ON THE PRODUCT OF LINEAR FORMS

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1. Introduction. We shall give in this note a new proof, differing in its essentials from any yet published, of Minkowski's

THEOREM. *If a_{ij} $(i, j=1, 2)$ and b_1, b_2 are real numbers, and if $D=|a_{ij}|\neq 0$, then there exist integers x_i such that*

$$|(a_{11}x_1 + a_{12}x_2 - b_1)(a_{21}x_1 + a_{22}x_2 - b_2)| \leqq \tfrac{1}{4}|D|. \tag{1}$$

Several proofs of this result have been published, and two proofs of its extension to $n=3$. References through 1936 are given in Koksma's "*Diophantische Approximationen*," Ergebnisse der Mathematik, 4, 1936, p. 19. A new proof for $n=2$ was given recently by Mordell (*J. London Math. Soc.*, 16, 1941, 86–88); and a very simple proof for $n=3$ was given by Davenport (ibid., 14, 1939, 47–51). The conjecture for any n is that if $L_1, \cdots, L_n$ are n real homogeneous linear forms in $x_1, \cdots, x_n$ with determinant D not zero, and $b_1, \cdots, b_n$ are n real numbers, then there exist integers x_i such that $\prod|L_i - b_i| \leqq 2^{-n}|D|$. No complete proof has yet been given for any $n \geqq 4$. However proofs have been given by Tschebotareff,* Siegel,† Davenport,† and Mordell[1] that there exists a constant γ_n, somewhat larger than 2^{-n}, which would serve in place of the 2^{-n}.

The generalization of our present proof seems to require a generalized euclidean algorithm, to be used in assuring that certain inequalities like our (12) shall work out successfully.

2. Permissible transformations. If any of the a_{ij} is zero, the theorem is immediate. For example if a_{12} is zero, then $D=a_{11}a_{22}\neq 0$, and we choose x_2 so that $|a_{22}x_2 - b_2| \leqq \tfrac{1}{2}|a_{22}|$, and for this x_2 choose x_1 so that $|a_{11}x_1 + a_{12}x_2 - b_1| \leqq \tfrac{1}{2}|a_{11}|$.

The following transformations of the forms $\phi_i = a_{i1}x_1 + a_{i2}x_2$ are *permissible*, provided the theorem can be proved for all b_i with respect to the derived pair:

* N. G. Tschebotareff, Sci. Notes Kazan Univ. 94, 14–16, 1934; and Vierteljschr. Naturforsch. Ges. Zürich, 85 Beiblatt, 27–30, 1940. Mordell's result is also in this Beiblatt, pp. 47–50.

† C. L. Siegel, in a letter to Mordell, October 10, 1937. At Siegel's suggestion, H. Davenport published a similar proof in the Acta Arithmetica, 2, 1937, 262–265.

(a) any transformation $x_i=t_{i1}y_1+t_{i2}y_2$ $(i=1, 2)$ of determinant ± 1 and integral coefficients t_{ij}; *e.g.* sign-changes and interchanges of the x_i;

(b) interchanging the forms ϕ_1 and ϕ_2;

(c) multiplying either form by a non-zero real number, *e.g.* -1.

LEMMA 1. *By permissible transformations we can replace* ϕ_1, ϕ_2 *by forms in which, either, one of the* a_{ij} *is zero, or*

$$a_{11} > 0, \quad a_{12} > 0, \quad a_{21} < 0, \quad a_{22} > 0. \tag{2}$$

Supposing no a_{ij} zero, we can obviously secure either (2) or

$$a_{ij} > 0 \qquad (i, j = 1, 2). \tag{3}$$

If (3) holds write $a_{12}=a_{11}q_1+a_{13}$, $a_{22}=a_{21}q_2+a_{23}$, where the q_i are integers, $0\leqq a_{13}<a_{11}$, $0\leqq a_{23}<a_{21}$. If now $q_1>q_2$ (and similarly if $q_1<q_2$), either some a_{ij} is zero or we obtain (2) on replacing x_1 by $x_1-q_1x_2$; for a_{11} and a_{21} are unaltered, $a_{12}\to a_{13}$, $a_{22}\to a_{23}-a_{21}(q_1-q_2)<0$. If $q_1=q_2$ we interchange columns and continue the process, thus obtaining for the q's the successive quotients in the continued fractions for a_{12}/a_{11} and a_{22}/a_{21}. The quotients must differ at some step unless $a_{12}/a_{11}=a_{22}/a_{21}$, that is $D=0$.

LEMMA 2. *If* (2) *holds a further transformation secures*

$$a_{11} > a_{12} > 0, \quad a_{22} > -a_{21} > 0. \tag{4}$$

A simple interchange and sign-change evidently allows us to assume $a_{11}>a_{12}>0$, $a_{21}<0$, $a_{22}>0$. If now $-a_{21}\geqq a_{22}$, let $-a_{21}=a_{22}q+r$, $0\leqq r<a_{22}$, $q\geqq 1$, and replace x_2 by x_2+qx_1. This replaces a_{11} by $a_{11}+qa_{12}$ *a fortiori* greater than a_{12}, and $-a_{21}$ by r.

To simplify our further argument we divide through by a_{11} and a_{22} respectively, and have

LEMMA 3. *In proving Theorem* 1 *we can restrict ourselves without loss of generality to the forms*

$$x+\rho y, \quad -\sigma x+y, \quad (0<\rho<1, 0<\sigma<1). \tag{5}$$

We shall presently require the additional condition $2\sigma\rho+\rho\geqq\sigma$, which can be achieved, if $\rho<\sigma$, by interchanging the forms, changing the signs of one form; and replacing x by y, y by $-x$.

3. The fundamental lemma. *If* $0<\rho<1$, $\sigma>0$, κ *and* λ *are real, and* $2\sigma\rho+\rho\geqq\sigma$, *then we can choose integers* x *and* y *to satisfy simultaneously*

$$-\tfrac{1}{2} < x+\rho y-\kappa \leqq \tfrac{1}{2}, \tag{6}$$

$$|(x+\rho y-\kappa)(-\sigma x+y-\lambda)| \leqq (1+\rho\sigma)/4. \tag{7}$$

There are solutions of (6) with y as small and x as large as we please, and hence such that $-\sigma x+y$ is arbitrarily small algebraically. For to any y, (6)

corresponds a unique x. For any solution (x_0, y_0) of (6) there is a unique r such that

$$(x_0,\ y_0 + i) \qquad (i = 0, \cdots, r) \tag{8}$$

are solutions of (6), but (x_0, y_0+r+1) is not. Abbreviating $\epsilon \equiv x_0+\rho(y_0+r)-\kappa$, we have

$$-\tfrac{1}{2} < \epsilon \leqq \tfrac{1}{2}, \qquad \epsilon + \rho > \tfrac{1}{2}. \tag{9}$$

Further, the values (8) give to $-\sigma x+y$ the values

$$-\sigma x_0 + y_0,\ -\sigma x_0 + y_0 + 1, \cdots, -\sigma x_0 + y_0 + r \equiv \eta,$$

and hence take care in respect to (7) of all values λ from a little less than $-\sigma x_0+y_0$ to $\eta+\mu$, where μ is defined by

$$|\epsilon| \cdot \mu = (1 + \rho\sigma)/4. \tag{10}$$

Next, (x_0-1, y_0+r+1) is likewise a solution of (6), giving to $x+\rho y-\kappa$ the value $\epsilon+\rho-1$, and to $-\sigma x+y$ the value $\eta+1+\sigma$; for by (9), $-\frac{1}{2}<\epsilon+\rho-1\leqq\frac{1}{2}$. These values take care of values λ from $\eta+1+\sigma-\nu$ to $\eta+1+\sigma+\nu$, where ν is defined by

$$|\epsilon + \rho - 1| \cdot \nu = (1 + \rho\sigma)/4. \tag{11}$$

A forward step will have been completed if $\mu+\nu\geqq 1+\sigma$, which can be expressed as follows:

$$\frac{1}{|\epsilon|} + \frac{1}{|\epsilon + \rho - 1|} \geqq \frac{4(1 + \sigma)}{1 + \rho\sigma}. \tag{12}$$

It remains only to verify that (12) holds in all cases.

I. *Case* $\epsilon<0$. Write $\theta=1-\epsilon$. Then by (5) and (9), $\rho<1<\theta<\rho+\frac{1}{2}$. Since the right hand side of (12) is $\leqq 4/\rho$ since $\rho<1$, (12) follows from $1/(\theta-1)+1/(\theta-\rho)\geqq 4/\rho$, and hence ($\theta$ being given its maximum value $\rho+\frac{1}{2}$) from

$$2/(2\rho - 1) + 2 \geqq 4/\rho, \qquad 4(\rho - 1)^2 \geqq 0. \tag{13}$$

II. *Case* $1-\rho<\epsilon\leqq\frac{1}{2}$. Then (12) holds if $1/\epsilon+1/(\epsilon+\rho-1)\geqq 4/\rho$, which reduces to (13) on giving ϵ its maximum value $\frac{1}{2}$.

III. *Case* $0<\epsilon<1-\rho$. Then (12) holds if

$$1/\epsilon + 1/(1 - \rho - \epsilon) \geqq 4(1 + \sigma)/(1 + \rho\sigma).$$

The left hand side is least when $\epsilon=(1-\rho)/2$, and the inequality reduces to $2\rho\sigma+\rho\geqq\sigma$.

A SURVEY COURSE FOR TEACHERS

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Much has been said and written concerning so called "survey" courses in high schools and colleges. In mathematics such courses are usually called General Mathematics and are offered to freshmen. In college, the students enrolled in these courses often have had meager high school training and in general are poorly prepared. They seem to possess neither the desire to broaden their mathematical horizon, nor the capacity for the proper appreciation. One noteworthy exception is the course on The Significance of Mathematics at the University of Buffalo.*

After teaching a general mathematics course for college freshmen with results that were none too encouraging we decided to experiment with a survey course for mathematics majors offered to seniors and students beginning graduate work. Consequently, in the summer session of 1941 we offered a course called "Mathematics for High School Teachers," its general aim or objective being stated in its description in the college catalogue: "Selected Topics from Higher Algebra and Geometry Designed to Strengthen Academically Teachers of High School Mathematics."

Recognizing that the study of mathematics begins somewhere in the "middle" and proceeds both toward its beginnings or foundations and toward abstract generalizations, an effort was made to take the class in both directions as far as time and abilities would warrant.

The outline for the course included the following topics:†

1. Definitions of mathematics.
2. Evolution of the number system: development of the number system from basic postulates; other number systems; elementary number theory.
3. Algebra and logic; Boolean algebra and other algebras.
4. Fundamental theorem of algebra.
5. Infinite series.
6. Criterion for ruler and compass constructions; trisection of an angle, duplication of the cube, squaring the circle.
7. Statistics, probability, interpolation, extrapolation.
8. Geometries: Lobachevskian, Riemannian, projective, modern.
9. The n nth roots and logarithms of complex numbers.
10. Meaning of topology.
11. Mathematics for recreation; mathematical puzzles.
12. Selected topics from the history of mathematics; famous mathematicians, historical and contemporary.

* Harriet F. Montague, A course on the significance of mathematics, this MONTHLY, vol. 48, 1941, pp. 681–684.

† Because of lack of time some of these had to be omitted. It is recommended that the topics omitted and the ones stressed be dependent upon the needs, desires and abilities of the individual students.

13. Applications of mathematics* to music, to physics, to biology, to agriculture, to industry; role of mathematics in modern warfare.

14. Relativity theory.

Such reference books as *Fundamentals of Mathematics* by Richardson and *Introduction to Mathematics* by Cooley, Gans, Kline, and Wahlert were quite valuable. However, these texts were written primarily for freshmen and therefore many omissions of elementary materials and supplementations of more advanced reading were necessary. For example, the materials concerning analytic geometry and calculus were omitted because the students had credits in these subjects; on the other hand proofs of the impossibility of the trisection of a given angle, *etc.*, were given in class, while these texts omit such proofs.

As the class progressed, many subjects for term papers and term reports presented themselves, the ideas for them growing out of topics briefly discussed in class. In this connection each individual was encouraged to select some phase of the work and explore it as far as time and facilities would permit as a sort of a miniature research project. Class time was given for reports on progress and many interesting bits of information concerning the history, philosophy, and applications of mathematics as well as pure mathematics were exchanged among the members. Although this definitely was not a methods course we felt free to connect certain contents of the course with their practical teaching problems. An effort was made to develop an awareness on the part of students to the possibility of a direct transfer to their own classroom situations.

As is so often the case, objective data concerning the lasting values which students obtain from a course are difficult to obtain. But in terms of subjective evaluation the results were quite pleasing. A high degree of interest was shown by each student. Some persons from other subject matter fields attended class regularly and took part in discussions without being enrolled for credit. Some students expressed disgust at the manner in which they had been teaching certain topics to high school classes because of "ignorance" on their part. One decided to experiment with a unit on Boolean algebra in her second year algebra class as a basis for a thesis topic. Another is planning some non-Euclidean experiments in plane geometry.

Obviously such a course as this might seem trivial for a university where research proper is stressed, but for a teachers college, the majority of whose students are not looking forward to the doctorate, it seems that a need is being met. At least, the results were pleasing enough to justify further experimentation with a survey course on an advanced level.

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* The class was encouraged to submit applications suitable for the inclusion in the 17th Yearbook of the National Council of Teachers of Mathematics.

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AN INDUCTIVE PROOF OF DESCARTES' RULE OF SIGNS

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1. Introduction. The so-called proofs of Descartes' Rule generally given in college algebras are merely verifications of special cases. The only rigorous direct proof in the literature known to me is that of L. E. Dickson's *First Course in the Theory of Equations*. This proof uses a rather complicated notation and has always seemed to me to be difficult to follow.

I have recently discovered an alteration in the proof which enables the omission of any consideration of permanences. This results in a fundamental simplification of the notation. I present it here in full detail.

2. Sequence of real numbers. Let n be a positive integer and consider a sequence

$$a_0, a_1, \cdots, a_n \tag{1}$$

of $n+1$ real numbers a_i. We shall be interested in pairs of *consecutive non-zero* terms a_i, a_j of our sequence, and shall call a_i and a_j *consecutive* either if $j=i+1$ or if $j>i+1$ but every intervening a_k, with $i<k<j$, is zero.

A *variation* in sign of our sequence is a pair of consecutive non-zero terms of opposite sign. Let us count the number of variations and designate the integer so obtained by

$$V(a_0, a_1, \cdots, a_n). \tag{2}$$

It is the number of changes of sign if we take as our first sign that of the first non-zero a_i (reading from left to right) and then pass to the right through all the signs of its non-zero terms. What then if the first and last non-zero terms have the same sign? In this case an even number of changes of sign must have

taken place, $V(a_0, a_1, \cdots, a_n)$ must be even. Similarly an odd number of changes of sign will result in a first and a last non-zero term which have opposite signs. We state a form of this result as

LEMMA 1. *Suppose that* $a_0a_n \neq 0$. *Then* $V(a_0, a_1, \cdots, a_n)$ *is even or odd according as* a_0a_n *is positive or negative.*

From this result we shall derive immediately

LEMMA 2. *Let* $r_0, \cdots, r_{n-1}$ *be a sequence of positive real numbers and derive a second sequence* $b_0, \cdots, b_n$ *from* (1) *by the definitions*

$$b_0 = a_0, \qquad b_j = a_j + r_{j-1}b_{j-1} \qquad (j = 1, \cdots, n). \tag{3}$$

Then if a_0, a_n, *and* b_n *are all not zero, and if* $V(a_0, \cdots, a_n) = V(b_0, \cdots, b_n)$, *the signs of* a_n *and* b_n *are the same.*

For our hypothesis and Lemma 1 imply that a_0a_n and a_0b_n have the same signs. Hence so do a_n and b_n.

If we adjoin a term a_{n+1} to our sequence we will have $V(a_0, a_1, \cdots, a_n) = V(a_0, a_1, \cdots, a_{n+1})$ either if $a_{n+1}=0$ or if the sign of a_{n+1} is the same as that of a_n. It is also clear that in all cases

$$V(a_0, a_1, \cdots, a_n) \leqq V(a_0, a_1, \cdots, a_{n+1}) \leqq V(a_0, a_1, \cdots, a_n) + 1. \tag{4}$$

We shall use these remarks in an inductive proof of our principal.

LEMMA 3. *Define the sequence* $b_0, b_1, \cdots, b_n$ *as in* (3). *Then* $V(b_0, b_1, \cdots, b_n) \leqq V(a_0, a_1, \cdots, a_n)$, *and if* a_0 *and* a_n *are not zero but* $b_n = 0$ *we have* $V(b_0, b_1, \cdots, b_n) < V(a_0, a_1, \cdots, a_n)$.

For if $n=1$ the number of variations in the sequence a_0, a_1 is zero or one according as a_0a_1 is not or is negative. Since $b_0b_1 = a_0a_1 + r_0a_0^2$ we see that if a_0a_1 is non-negative so is b_0b_1, the number of variations of sign is zero in both cases. Moreover if $a_0 \neq 0$ then b_0b_1 is positive, $b_1 \neq 0$. If a_0a_1 is negative then $V(a_0, a_1) = 1 \geqq V(b_0, b_1)$ and $V(a_0, a_1) > V(b_0, b_1) = 0$ if $b_1 = 0$.

Assume now that our result is true for all pairs of related sequences of our type with $n=m$, and consider two sequences $a_0, a_1, \cdots, a_{m+1}$, and $b_0, b_1, \cdots, b_{m+1}$. If $a_0 = 0$ then $b_0 = 0$, $b_1 = a_1 + r_0a_0 = a_1$ the sequences $a_1, a_2, \cdots, a_{m+1}$ and $b_1, b_2, \cdots, b_{m+1}$ have precisely the same numbers of variations in sign as our original two sequences and are related sequences for $n = m$. Our conclusion then follows from the hypothesis of our induction. If some other $a_j = 0$ then $b_j = a_j + r_{j-1}b_{j-1} = r_{j-1}b_{j-1}$ has the same sign as b_{j-1} and we may delete a_j from our first sequence and b_j from our second sequence without changing the number of variations of sign in either case. Moreover the new sequences are related in the prescribed fashion since $b_{j+1} = a_{j+1} + r_jb_j = a_{j+1} + (r_{j-1}r_j)b_{j-1}$. Hence our result follows again from the hypothesis of our induction.

There remains only the case where no one of the numbers $a_0, a_1, \cdots, a_{m+1}$ is zero. We note the trivial relations

$$(5) \qquad V(b_0, b_1, \cdots, b_m) \leqq V(a_0, a_1, \cdots, a_m) \leqq V(a_0, a_1, \cdots, a_{m+1}).$$

If $b_{m+1}=0$ then the hypothesis $V(b_0, b_1, \cdots, b_m) < (a_0, a_1, \cdots, a_m)$ implies that $V(b_0, b_1, \cdots, b_{m+1}) < V(a_0, a_1, \cdots, a_{m+1})$. However if $V(b_0, b_1, \cdots, b_m) = V(a_0, a_1, \cdots, a_m)$ the hypothesis of our induction implies that $b_m \neq 0$ and has the same sign as a_m by Lemma 2. But then $b_{m+1}=a_{m+1}+r_m b_m=0$ only if a_{m+1} and b_m have opposite signs. Then a_{m+1} and a_m will have opposite signs, the sequence $a_0, a_1, \cdots, a_{m+1}$ has one more variation in sign than the sequence $a_0, a_1, \cdots, a_m$ and one more than $b_0, b_1, \cdots, b_{m+1}$ as desired. Finally let $b_{m+1} \neq 0$. From (4) if $V(b_0, b_1, \cdots, b_m) < V(a_0, a_1, \cdots, a_m)$ then we have our desired inequality. The only possibility for it not to hold is indeed when $V(b_0, b_1, \cdots, b_m) = V(a_0, b_1, \cdots, a_m)$, when a_m and a_{m+1} have the same signs, and when b_m and b_{m+1} have opposite signs. But by Lemma 2 a_m and b_m have the same signs and if $b_m b_{m+1}=b_m^2 r_m+a_{m+1}b_m$ is negative so is $a_{m+1}b_m$. Then $a_{m+1}a_m$ is negative. This proves the lemma.

3. Polynomials with real coefficients. We shall assume, as is usual, the analytic result stating that if $f(x)$ is a polynomial with real coefficients and $a<b$ then there is an odd number or an even number of real roots in the interval $a<x<b$ according as $f(a)\cdot f(b)$ is negative or positive. We then have

LEMMA 4. *Let $a_0, \cdots, a_n$ be real numbers such that $a_0 a_n \neq 0$, and $f(x)=a_0x^n + \cdots + a_n$. Then $f(x)$ has either an odd or an even number of positive roots according as $a_0 a_n$ is negative or positive.*

For the sign of a_0a_n is the same as that of $c_n=a_0^{-2}(a_0a_n)=a_0^{-1}a_n$. But c_n is the constant term of $\phi(x)=a_0^{-1}f(x)=x^n+c_1x^{n-1}+\cdots+c_n$. This polynomial has the same roots as $f(x)$ and $\phi(0)=c_n$, $\phi(h)>0$ for $h>g$ where g may actually be taken to be any number greater than all the numbers $1+|c_j|$. Then the positive roots of $f(x)$ lie in the interval $0<x<h$ and, by the result assumed, there is an odd number of such roots if $c_n>0$, an even number if $c_n<0$.

We now arrive quickly at

DESCARTES' RULE OF SIGNS. *Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ have real coefficients and t be the number of positive roots of $f(x)=0$. Then the difference*

$$V(a_0, a_1, \cdots, a_n) - t,$$

is a non-negative even integer.

We may clearly assume that a_0 and a_n are not zero. If ρ is a positive real root of $f(x)$ we may write $f(x)=(x-\rho)\phi(x)$ where $\phi(x)=b_0x^{n-1}+\cdots+b_{n-1}$, the b_i are computed as in (3) with the $r_j=\rho$, $b_n=0$. By Lemma 3 we have $V(b_0, \cdots, b_{n-1}) \leqq V(a_0, a_1, \cdots, a_n)-1$. After t such steps we obtain $f(x)=(x-\rho_1)\cdots(x-\rho_t)\psi(x)$, where $\psi(x)=c_0x^{n-t}+c^{n-t-1}+\cdots+c_{n-t}$, $0 \leqq V(c_0, c_1, \cdots, c_{n-t}) \leqq V(a_0, a_1, \cdots, a_n)-t$. Hence $V(a_0, a_1, \cdots, a_n) \geqq t$ as desired. The evenness of their difference follows from the fact that the criteria for evenness of t and $V(a_0, a_1, \cdots, a_n)$ in Lemmas 4 and 1 are the same.

DISTRIBUTION OF POINTS IN n-SPACE

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1. Introduction. Numerous writers have been concerned with the relation satisfied by the $(n+1)(n+2)/2$ mutual (positive) distances of $n+2$ pairwise distinct points of a euclidean n-space E_n. Known to Lagrange for the case of five points p_1, p_2, p_3, p_4, p_5 in E_3, this relation was established anew by Cayley in his first published paper and exhibited in determinant form

$$D(p_1, p_2, p_3, p_4, p_5) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_1p_2^2 & \cdots & p_1p_5^2 \\ 1 & p_2p_1^2 & 0 & \cdots & p_2p_5^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & p_5p_1^2 & p_5p_2^2 & \cdots & 0 \end{vmatrix} = 0,$$

(where $p_ip_j = p_jp_i$ denotes the distance of the points p_i, p_j) together with the relations $D(p_1, p_2, p_3) = 0$, $D(p_1, p_2, p_3, p_4) = 0$ satisfied by the mutual distances of three and four points in a line and plane, respectively.*

If $V(p_0, p_1, \cdots, p_k)$ denotes the volume of the euclidean simplex $\{p_0, p_1, \cdots, p_k\}$, Cayley's method is readily extended to yield

$$V^2(p_0, p_1, \cdots, p_k) = \frac{(-1)^{k+1}}{2^k(k!)^2} D(p_0, p_1, \cdots, p_k);$$

and hence if $n+2$ points $p_0, p_1, \cdots, p_{n+1}$ are in E_n, their determinant $D(p_0, p_1, \cdots, p_{n+1})$ clearly vanishes since the volume of the *degenerate* simplex $\{p_0, p_1, \cdots, p_{n+1}\}$ is evidently zero. This formula gives an additional important relation which is not mentioned by some of the early writers; for as $V^2(p_0, p_1, \cdots, p_{n+1})$ is non-negative, it follows that the determinant D formed for *any euclidean m-tuple either vanishes or has the sign of* $(-1)^m$. Since these facts are basic in Menger's characterization of the euclidean metric among all semimetrics (the structure of the distance matrix (p_ip_j) of a semimetric m-tuple which is congruently imbeddable in a euclidean space being described in Menger's development in terms of the signs of the determinants D formed for subsets of the m-tuple) we are prompted to refer to D as the Cayley-Menger determinant.†

It is noted that any *bordered* principal minor of $D(p_0, p_1, \cdots, p_{n+1})$ is itself a Cayley-Menger determinant $D(p_{i_0}, p_{i_1}, \cdots, p_{i_k})$ and hence (for $p_0, p_1, \cdots, p_{n+1}$ a euclidean $(n+2)$-tuple) either vanishes or has the sign of $(-1)^{k+1}$. Further, it is well-known that the *unbordered* principal minor $C(p_0, p_1, \cdots, p_{n+1}) = |p_ip_j^2|$,

* A. Cayley, On a theorem in the geometry of position, Cambridge Mathematical Journal, 2, 1841, pp. 267–271; Collected Papers, vol. 1, pp. 1–4. See also J. J. Sylvester, On Staudt's theorems concerning the contents of polygons and polyhedrons, Collected Works, pp. 382–391.

† See Chapter III of L. M. Blumenthal, Distance Geometries, University of Missouri Studies, vol. 13, 1938.

$(i, j=0, 1, \cdots, n+1)$, has for a euclidean $(n+2)$-tuple the sign of $(-1)^{n+1}$ or vanishes, with the latter case holding if and only if $p_0, p_1, \cdots, p_{n+1}$ are on a hyperspherical surface $S_{n-1,r}$ (which may, in particular, be a hyperplane E_{n-1}). Hence the classical theory interprets geometrically the signs of the cofactors of all elements in the *principal diagonal* of the Cayley-Menger determinant of $n+2$ points in E_n.

The purpose of this note is to complete the theory by exhibiting the geometrical significance of the signs of the cofactors of the remaining elements of the determinant $D(p_0, p_1, \cdots, p_{n+1})$, where, in order not to unduly complicate the discussion, we suppose this euclidean $(n+2)$-tuple not to lie in a hyperplane E_{n-1}. The results, obtained without the use of coördinates, are believed to be new. They lead to a complete description of the distribution of $n+2$ points in E_n in terms of the mutual distances of the points. We offer, further, an application of the results, and the statement of analogous theorems proved for $(n+2)$-tuples in hyperbolic and spherical n-dimensional spaces.

2. Cofactors of bordering elements. We have to examine cofactors of the elements 1 in the bordering row and column of D (since we are concerned with non-principal cofactors) and take the cofactor of the element in the first row and last column as typical. Denote this cofactor by $\gamma(p_0, \cdots, p_n; p_{n+1})$.

THEOREM 2.1. *Let $\gamma(p_0, \cdots, p_n; p_{n+1})$ be different from zero. Then* (1) *the points $p_0, p_1, \cdots, p_n$ are in E_n, not in E_{n-1}, and* (2) *p_{n+1} is outside or inside the sphere circumscribing $p_0, p_1, \cdots, p_n$ if and only if the sign of $\gamma(p_0, \cdots, p_n; p_{n+1})$ is $(-1)^n$ or $(-1)^{n+1}$, respectively.*

Proof. Since $D(p_0, p_1, \cdots, p_{n+1})=0$, we have

$$(*) \quad D(p_0, p_1, \cdots, p_{n-1}, p_n)\cdot C(p_0, p_1, \cdots, p_n, p_{n+1}) - \gamma^2(p_0, \cdots, p_n; p_{n+1}) = 0,$$

and as $\gamma(p_0, p_1, \cdots, p_n; p_{n+1}) \neq 0$ neither of the determinants in the first term of (*) vanishes. Hence the points $p_0, p_1, \cdots, p_{n-1}, p_n$ are not in E_{n-1} and the $n+2$ points are not on a hypersphere $S_{n-1,r}$.

Denote by o the center of the circumsphere of $p_0, p_1, \cdots, p_{n-1}, p_n$ and let r be its radius. Subtracting from the last row (column) of $D(p_0, p_1, \cdots, p_{n+1}, o)$ the product of the first row (column) by r^2 and applying the Cauchy development to the resulting determinant gives

$$(op_{n+1}^2 - r^2)^2 D(p_0, \cdots, p_n) + 2(op_{n+1}^2 - r^2)\gamma(p_0, \cdots, p_n; p_{n+1}) + C(p_0, \cdots, p_{n+1}) = 0.$$

Substituting for $C(p_0, p_1, \cdots, p_{n+1})$ its value from (*), and recognizing that $op_{n+1}^2 - r^2$ is the power $P(p_{n+1})$ of p_{n+1} with respect to the circumsphere, we obtain

$$P(p_{n+1}) = -\gamma(p_0, \cdots, p_n; p_{n+1})/D(p_0, \cdots, p_n).$$

Since sgn $D(p_0, \cdots, p_n)=(-1)^{n+1}$, $P(p_{n+1})$ is positive or negative if and

only if sgn $\gamma(p_0, \cdots, p_n; p_{n+1})$ is $(-1)^n$ or $(-1)^{n+1}$, respectively, and the theorem is proved.

In case $\gamma(p_0, \cdots, p_n; p_{n+1})=0$ then either $\mathcal{D}(p_0, \cdots, p_{n-1}, p_n)$ or $C(p_0, \cdots, p_n, p_{n+1})$ vanishes. In the latter case the $n+2$ points are on a sphere $S_{n-1,r}$, which might be degenerate (*i.e.*, a hyperplane E_{n-1}), while in the former case (with $C(p_0, \cdots, p_n, p_{n+1})\neq 0$) the $n+1$ points $p_0, p_1, \cdots, p_{n-1}, p_n$ are in an E_{n-1} which, according to our assumption on the $(n+2)$-tuple, does not contain p_{n+1}. These $n+1$ points may be examined in the light of the above and later results.

3. Cofactors of non-bordering elements. We are concerned here with cofactors $[p_ip_j^2]$ of elements $p_ip_j^2$ $(i\neq j)$, of D. Selecting the cofactor of $p_np_{n+1}^2$ as typical, we have

THEOREM 3.1. *Let* $[p_np_{n+1}^2]$ *be different from zero. Then* (1) *the points* $p_0, p_1, \cdots, p_{n-1}$ *determine an* $(n-1)$*-dimensional hyperplane and* (2) p_n *and* p_{n+1} *are on the same side or on opposite sides of this hyperplane if and only if the sign of* $[p_np_{n+1}^2]$ *be* $(-1)^n$ *or* $(-1)^{n+1}$, *respectively.*

Proof. Since $D(p_0, \cdots, p_n, p_{n+1})=0$ we have

$$(**) \qquad D(p_0, \cdots, p_n)\cdot D(p_0, \cdots, p_{n-1}, p_{n+1})-[p_np_{n+1}^2]^2=0.$$

The non-vanishing of $[p_np_{n+1}^2]$ implies that neither of the determinants whose product forms the first term in (**) vanishes, and hence the points $p_0, p_1, \cdots, p_{n-1}$ are in E_{n-1}, not in E_{n-2}, and determine a hyperplane $\pi(p_0, \cdots, p_{n-1})$ which does not contain either p_n or p_{n+1}.

Replacing p_{n+1} in (**) by any point x of E_n, it is seen that $[p_nx^2]$ vanishes if and only if x is a point of π and hence sgn $[p_np^2]=$sgn $[p_nq^2]$ *for any two points* p, q *on the same side of* π.

If, now, p_n and p_{n+1} are on the same side of π then sgn $[p_np_{n+1}^2]=$sgn $[p_np_n^2]$, and since $[p_np_n^2]=-D(p_0, \cdots, p_n)$ we have in this case sgn $[p_np_{n+1}^2]=(-1)^n$.

On the other hand, suppose that p_n and p_{n+1} are on opposite sides of π, and denote the reflection of p_n in π by p_n^*. Then sgn $[p_np_{n+1}^2]=$sgn $[p_np_n^{*2}]$. Inspection of the vanishing determinant $D(p_0, \cdots, p_n, p_n^*)$ gives

$$(***) \qquad [p_np_n^{*2}] = -\{D(p_0, \cdots, p_{n-1}, p_n)+(p_np_n^{*2})\cdot D(p_0, \cdots, p_{n-1})\},$$

while from (**) we have

$$[p_np_n^{*2}]^2 = D^2(p_0, \cdots, p_{n-1}, p_n).$$

It follows that $[p_np_n^{*2}]=D(p_0, \cdots, p_{n-1}, p_n)$ and hence sgn $[p_np_n^{*2}]=(-1)^{n+1}$. Thus for p_n and p_{n+1} on opposite sides of π sgn $[p_np_{n+1}^2]=(-1)^{n+1}$.

To establish the converse it suffices to show that if sgn $[p_np_{n+1}^2]=(-1)^n$ then p_n, p_{n+1} are on the same side of π. This is immediate, for then sgn $[p_np_{n+1}^2]\neq$sgn $[p_np_n^{*2}]$ and hence p_n^*, p_{n+1} are not on the same side of π. Since none

of the points p_n, p_{n+1}, p_n^* lies in π, it follows that p_n and p_{n+1} are on the same side of π, and the theorem is proved.

COROLLARY. *The point p_{n+1} is inside the simplex with vertices $p_0, p_1, \cdots, p_n$ if and only if each cofactor $[p_i p_{n+1}^2]$, $(i=0, 1, \cdots, n)$, in $D(p_0, \cdots, p_{n+1})$ has the sign of $(-1)^n$.*

Theorems 2.1 and 3.1 may be combined to characterize the regions bounded by $S_{n-1,r}$ and the "faces" of an inscribed simplex.

4. An application. In an article dealing with pseudo-planar quintuples it was found necessary to establish the fact that if four points are in a plane at least one of them is inside or on the circumcircle of the other three.* As an immediate application of Theorem 2.1 we have

THEOREM 4.1. *If $p_0, p_1, \cdots, p_n, p_{n+1}$ are $n+2$ points of E_n at least one of these points is inside or on an $(n-1)$-dimensional spherical surface $S_{n-1,r}$ passing through the others.*

Proof. If the contrary be supposed then each of the points $p_0, p_1, \cdots, p_{n+1}$ is outside any $S_{n-1,r}$ containing the remaining $n+1$ points. It follows from Theorem 2.1 that each of the cofactors $\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)$, $(i=0, 1, \cdots, n+1)$, has the sign of $(-1)^n$ whenever it does not vanish. Since

$$\sum_{i=0}^{n+1} \gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i) = D(p_0, \cdots, p_{n+1}) = 0,$$

all of these cofactors must vanish. But this implies that either $C(p_0, \cdots, p_{n+1}) = 0$ (and hence the $n+2$ points are on an $S_{n-1,r}$) or $D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}) = 0$, $(i=0, 1, \cdots, n+1)$, and the $n+2$ points are in an E_{n-1} (a degenerate $S_{n-1,r}$). This contradiction yields the theorem.

Closely connected with this is the following theorem (well-known at least for the case $n=3$);†

THEOREM 4.2. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ non-cospherical points of E_n, with no $(n+1)$-tuple in an E_{n-1}. Then the sum of the reciprocals of the power of each point with respect to the sphere circumscribing the remaining $n+1$ points is zero.*

We propose to give a new proof of this theorem, utilizing the expression for the power of a point developed in Theorem 2.1.‡

Since the $n+2$ points are non-cospherical, $C(p_0, \cdots, p_{n+1})$ is not zero, and since no $n+1$ of the points are in an E_{n-1}, each $(n+1)$-tuple has a non-vanishing determinant D. It follows that each of the cofactors

* P. M. Pepper, Concerning pseudo-planar quintuples, Reports of a Mathematical Colloquium, No. 2, 1940, p. 28–32.

† See, for example J. L. Coolidge, Treatise on the Circle and Sphere, p. 254. The reader is there asked to prove the theorem for 5 points in E_3 by applying the Darboux-Frobenius Identity.

‡ It is worth remarking that for planar quadruples a stronger theorem is valid, for there does not exist a plane set p_0, p_1, p_2, p_3 with p_3, p_2, p_1 inside the circumcircles of p_0, p_1, p_2; p_0, p_1, p_3; p_0, p_2, p_3, respectively.

$\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)$, $(i=0, 1, \cdots, n+1)$, in $D(p_0, \cdots, p_n, p_{n+1})$ is different from zero, and that the power $P(p_i)$ with respect to the sphere circumscribing $p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}$ is, from Theorem 2.1, given by

$$P(p_i) = -\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)/D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}).$$

Since

$$D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}) \cdot C(p_0, \cdots, p_{n+1}) - \gamma^2(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i) = 0,$$

we have

$$\begin{aligned}\sum_{i=0}^{n+1} 1/P(p_i) &= [-1/C(p_0, \cdots, p_{n+1})] \cdot \sum_{i=0}^{n+1} \gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i), \\ &= -D(p_0, \cdots, p_n, p_{n+1})/C(p_0, \cdots, p_{n+1}), \\ &= 0,\end{aligned}$$

and the theorem is proved.

5. Analogous theorems for spherical and hyperbolic spaces. If $p_0, p_1 \cdots, p_{n+1}$ are $n+2$ points of the n-dimensional spherical surface $S_{n,r}$ (the "surface" of a sphere of radius r in euclidean $(n+1)$-space, with geodesic (shorter arc) metric), then it is well-known that the determinant $\Delta(p_0, \cdots, p_{n+1}) = |\cos(p_ip_j/r)|$ $(i, j=0, 1, \cdots, n+1)$, vanishes, and has each principal minor non-negative. The non-vanishing of a principal minor $\Delta(p_{i_0}, p_{i_1}, \cdots, p_{i_k})$ signifies that the $k+1$ points $p_{i_0}, \cdots, p_{i_k}$ form an independent set (*i.e.*, they do not lie in a great hypersphere $S_{k-1,r}$. Concerning the geometrical significance of the signs of the non-principal minors, we have established the following theorem:

THEOREM 5.1. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of $S_{n,r}$ with the cofactor $[\cos(p_np_{n+1}/r)]$ of the element $\cos(p_np_{n+1}/r)$ in $\Delta(p_0, \cdots, p_{n+1})$ not zero. Then (1) the points $p_0, p_1, \cdots, p_{n-1}$ determine an $(n-1)$-dimensional great hypersphere $S_{n-1,r}$ and (2) the points p_n and p_{n+1} lie on the same or on opposite sides of $S_{n-1,r}$ if and only if $[\cos(p_np_{n+1}/r)]$ be negative or positive, respectively.*

As an obvious corollary we have a metric characterization of the interior of a spherical simplex.

Denoting the n-dimensional hyperbolic space of space constant μ by $H_{n,\mu}$, we can establish

THEOREM 5.2. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of $H_{n,\mu}$ with the cofactor $[\cosh(p_np_{n+1}/\mu)]$ of the element $\cosh(p_np_{n+1}/\mu)$ in the determinant $|\cosh(p_ip_j/\mu)|$, $(i, j=0, 1, \cdots, n+1)$, not zero. Then (1) the points $p_0, p_1, \cdots, p_{n-1}$ determine an $(n-1)$-dimensional hyperbolic hyperplane $H_{n-1,\mu}$ and (2) the points p_n, p_{n+1} are on the same or on opposite sides of $H_{n-1,\mu}$ if and only if $[\cosh(p_np_{n+1}/\mu)]$ has the sign of $(-1)^{n+1}$ or $(-1)^n$, respectively.*

These theorems furnish the basis for a metric characterization of euclidean, spherical, and hyperbolic simplices among all semimetric spaces.

THE CURTAIN ROD PROBLEM

W. A. BLANKINSHIP, University of Virginia

The following problem was proposed by Professor E. J. McShane: "Determine the shape that a rod of uniform cross-section, density, and elasticity will assume if forced to pass through the three non-collinear points, $(a, 0)$, $(-a, 0)$, and $(0, b)$."

This problem, as the name suggests, arose from trying to bend a curtain rod across an arched window. It is of interest in that it is one of the few calculus of variations problems arising from physical circumstances which has a solution in terms of well-known and tabulated functions. Note that the problem is phrased so as not to imply a fixed length of the rod, otherwise there would arise forces of compression which would considerably disturb the assumptions we are about to make.

The situation, then, is this. In whatever shape the rod happens to be, there will be a certain amount of potential energy stored in the rod due to its bending. Thus the rod will assume the shape for which the least amount of potential energy is stored, subject to the conditions that it still pass through the required points. It can be shown* that the potential energy stored in the rod when it assumes a given shape is proportional to the integral with respect to arc length of the square of the curvature, the constant of proportionality being determined by the moment of inertia of a cross-section and by the elasticity of the rod. Thus, in seeking to minimize a certain integral, we have a calculus of variations problem.

It can also be seen from physical considerations that the minimizing curve will be symmetric to the y-axis. Thus we need only consider the right-hand part of the curve, replacing the left-hand end condition by the condition that the tangent to the curve at $(0, b)$ be parallel to the x-axis.

At this point we could set up the problem in non-parametric form, assuming the equation of the curve to be given in the form, $y = y(x)$. However, this straightforward procedure leads to Euler equations which are rather difficult to handle. Let us then take arc length, s, as an independent variable, and replace the symbols x, y by y_1, y_2 respectively. The curve will then be given by equations of the form:

$$y_i = y_i(s) \qquad i = 1, 2; \; s_1 \leqq s \leqq s_2$$

with

$$(y_1')^2 + (y_2')^2 - 1 = 0.$$

The curvature of the curve will then be given by

$$K = y_1' y_2'' - y_1'' y_2',$$

and the integral of the square of this function is that which we wish to minimize. To treat this integrand we introduce the two new variables, y_3 and y_4, in the places of y_1' and y_2', resp. This substitution makes our integrand become

* See Lord Rayleigh's "Theory of Sound," vol. I, p. 255.

$$K^2 = (y_1' y_4' - y_2' y_3')^2$$

and introduces the differential equations

$$y_1' - y_3 = 0 \quad \text{and} \quad y_2' - y_4 = 0$$

to be satisfied.

Thus we have a Lagrange problem. Although it appears that we are using a parametric formulation, we must remember that s is arc length and is to be regarded as independent variable. Moreover our differential equations are not homogeneous and would not retain validity under change of parameter. We shall assume that arc length is measured from the left-hand end point.

Summing up the problem, then, we have the following: We wish to minimize the integral,

$$\int_{s_1}^{s_2} (y_1' y_4' - y_2' y_3')^2 ds,$$

in the class of C^2 curves,

$$y_i = y_i(s) \qquad i = 1, 2, 3, 4; \; s_1 \leqq s \leqq s_2,$$

satisfying the differential equations,

$$\phi_1 \equiv (y_1')^2 + (y_2')^2 - 1 = 0,$$
$$\phi_2 \equiv y_1' - y_3 = 0,$$
$$\phi_3 \equiv y_2' - y_4 = 0,$$

and the end conditions,

$$s_1 = y_1(s_1) = y_2(s_1) - b = y_3(s_1) - 1 = y_4(s_1) = 0,$$
$$y_1(s_2) - a = y_2(s_2) = 0.$$

Then the Lagrange multiplier rule states that there exist functions, $\lambda_0(s) = \text{const.} \geqq 0$, $\lambda_1(s)$, $\lambda_2(s)$, and $\lambda_3(s)$, not all zero, such that if we define

$$F(y, y', \lambda) \equiv \lambda_0 K^2 + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3,$$

then on the minimizing set, $y_i^0(s)$, the function F satisfies the Euler equations

$$\frac{d}{ds} F_{y_i'} - F_{y_i} = 0, \qquad i = 1, 2, 3, 4$$

$$\frac{d}{ds} (F - y_i' F_{y_i'}) - F_s = 0,$$

and the transversality conditions,

(1) $$F_{y_3'}(s_2) = 0$$

(2) $$F_{y_4'}(s_2) = 0$$

(3) $$F(s_2) - y_i' F_{y_i'}(s_2) = 0.$$

Hereafter we shall omit the superscript from y_i^0, and this shall be understood to be the minimizing curve.

Since F_{y_1}, F_{y_2}, and F_s are all zero, the first two and the last of the Euler equations are immediately integrable, and our complete set is:

$$2\lambda_0 K y_4' + 2\lambda_1 y_1' + \lambda_2 = 2c_1, \tag{4}$$

$$-2\lambda_0 K y_3' + 2\lambda_1 y_2' + \lambda_3 = 2c_2, \tag{5}$$

$$\frac{d}{ds}(-2\lambda_0 K y_2') = -\lambda_2, \tag{6}$$

$$\frac{d}{ds}(2\lambda_0 K y_1') = -\lambda_3, \tag{7}$$

$$F - y_i' F_{y_i'} = \text{const.} \tag{8}$$

Now we note that if λ_0 is zero, so are λ_2 and λ_3, in which case the solution to the Euler equations is a straight line, which is impossible unless $(0, b)$ is the origin. Thus we may assume that λ_0 is 1. Substituting the values for λ_2 and λ_3 from (6) and (7) into (4) and (5), we get

$$2Ky_4' + \lambda_1 y_1' + K' y_2' = c_1, \tag{9}$$

$$-2Ky_3' + \lambda_1 y_2' - K' y_1' = c_2. \tag{10}$$

Multiplying (9) and (10) by y_2' and y_1' respectively, and subtracting, and making use of the equations,

$$\phi_1 = \phi_2 = \phi_3 = 0,$$

and also

$$\tfrac{1}{2}\phi_1' = y_1' y_3' + y_2' y_4' = 0,$$

we obtain

$$K' = c_1 y_2' - c_2 y_1', \tag{11}$$

which integrates to

$$K = c_1 y_2 - c_2 y_1 + d. \tag{12}$$

Now from (3) and (8) we have

$$F - y_i' F_{y_i'} \equiv 0$$

or

$$4K^2 + 2\lambda_1(y_1'^2 + y_2'^2) + \lambda_2 y_1' + \lambda_3 y_2' - F = 0. \tag{13}$$

But from (4) and (5)

$$2\lambda_1(y_1'^2 + y_2'^2) + \lambda_2 y_1' + \lambda_3 y_2' = 2c_1 y_1' + 2c_2 y_2' - 2K^2,$$

which with (13) and $\phi_1 = \phi_2 = \phi_3 = 0$ gives

$$K^2 + 2c_1 y_1' + 2c_2 y_2' = 0. \tag{14}$$

From (11) and (14) we can eliminate y_1' and y_2' and we get the differential equation:

$$K^4 + 4K'^2 = 4(c_1^2 + c_2^2) \equiv \alpha^2,$$

which may be written

$$\tfrac{1}{2}ds = \frac{dK}{\sqrt{\alpha^2 - K^4}}.$$

This equation can be integrated and we get

$$\tfrac{1}{2}(s - \sigma)\sqrt{2\alpha} = cn^{-1}(K/\sqrt{\alpha}, \tfrac{1}{2}\sqrt{2})$$

or

$$K = \sqrt{\alpha}\, cn(\sqrt{\tfrac{1}{2}\alpha}(s - \sigma), \tfrac{1}{2}\sqrt{2})$$

where σ is a constant of integration.

We also need to evaluate

$$2\psi(s) \equiv \int^{s} - K^2 ds = -2\sqrt{2\alpha}\, E(\sqrt{\tfrac{1}{2}\alpha}(s - \sigma), \tfrac{1}{2}\sqrt{2}) + \alpha(s - \sigma),$$

where E is defined by

$$E(u, k) \equiv \int_0^u [1 - k^2 sn^2(u, k)]du.$$

Thus equations (12) and (14) become

$$c_1 y_2 - c_2 y_1 + d = K(s) \tag{15}$$

$$c_1 y_1 + c_2 y_2 + e = \psi(s). \tag{16}$$

These two equations may now be solved for y_1 and y_2 explicity in terms of s, and moreover all the functions involved are well-tabulated. The transversality conditions, (1) and (2), are

$$-2K(s_2)y_2'(s_2) = 0 \quad \text{and} \quad 2K(s_2)y_1'(s_2) = 0,$$

which, with

$$y_1'^2 + y_2'^2 = 1,$$

yields $K(s_2) = 0$.

Thus the transversality condition, $K(s_2) = 0$, and the end conditions,

$$-K^2(0) = 2c_1 \qquad \text{from (14)}$$

$$K(0) = bc_1 + d,\ K(s_2) = -ac_2 + d \qquad \text{from (12)}$$

$$\psi(s_2) = ac_1 + e,\quad \psi(0) = bc_2 + e \qquad \text{from (16)}$$

together with the equations (15) and (16) constitute a complete solution of the problem

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

SQUARE ROOTS FROM A TABLE OF COSINES

W. S. H. CRAWFORD, University of Minnesota

The three real roots of the reduced cubic equation

$$y^3 + py + q = 0 \qquad (p^3/27 + q^2/4 < 0)$$

are usually found* by making the transformation $y = nz$ and then comparing the resulting equation

$$z^3 + \frac{p}{n^2} z + \frac{q}{n^3} = 0$$

with the trigonometric identity

$$\cos^3 A - \tfrac{3}{4} \cos A - \frac{\cos 3A}{4} = 0.$$

This same scheme of relating an algebraic equation to a trigonometric identity, when applied to the quadratic equation $x^2 = k$, gives a method for finding the square root of a positive real number by means of a little mental arithmetic and a table of cosines. We let $x = ny$, where n is a positive number, and compare $y^2 = k/n^2$ with the trigonometric identity

$$\cos^2 A = (1 + \cos 2A)/2.$$

We evidently wish to take

$$k/n^2 = (1 + \cos 2A)/2,$$

which gives

$$\cos 2A = 2k/n^2 - 1. \tag{1}$$

We will then have $y = \cos A$, or

$$x = n \cos A. \tag{2}$$

In order for an angle A to exist such that (1) holds it is necessary and sufficient that $2k/n^2 - 1 \leqq 1$, which gives

$$n \geqq \sqrt{k}. \tag{3}$$

* For example, see Dickson, First Course in the Theory of Equations, p. 49.

This is the only restriction on n; in the case of the cubic, n is uniquely determined. However, a large value of n will cause a correspondingly large error in the determination of x from (2), and so it is best to use the smallest convenient value of n satisfying (3). The computations are simplest when n is taken to be a power of 10. Some sample computations, to four places, are given below.

k	n	$2k/n^2-1$	$2A$	A	$n\cos A$
8739	100	.7478	41°36′	20°48′	93.48
.3661	1	−.2678	105°32′	52°46′	.6051
1.21	10	−.9758	167°22′	83°41′	1.100

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

Mathematics Club of Washington Square College, New York University

The club held regular meetings throughout the year on the average of one every two weeks. The lectures on the following topics were all well attended: *Mathematics for defense* by Professor Frederick W. John, *Non-euclidean geometry* by Dr. Morris Kline, *Geometrical transformations* or *Pushing points around* by Mr. David Gans, *Insurance* by Mr. Howard Wahlert, *Vector analysis* by Mr. John Rohrbaugh, *Actuarial mathematics* by Professor Tilley, *Mathematical orientation with respect to scientific methodology* by Professor Thomas Jenkins, *Non-dimensional variables* by Mr. Melvin Lax. Four socials were held during the year, including the annual boat ride. At one of these the winners of the Freshman Mathematics Contest, selected from a field of over fifty entrants, were awarded their prizes—a $25 defense bond to the winner, Mr. William Sollfrey, and copies of *What is mathematics* by Courant and Robbins to Miss Marion King, Mr. Joseph Allison, and Mr. Harvey Goldstein, the runners up. In the William Lowell Putnam contest, the New York University team composed of Mr. Melvin Lax, Mr. Henry Shenker, and Mr. Harold Lewis received honorable mention. The club publication, *Math X*, was completely sold out in an edition of 400 copies. Officers for 1941–42 and for 1942–43 were as follows. President, Mr. Melvin Lax (41–42), Mr. Marvin Forray (42–43); Vice-President, Miss Norma Ornstein (41–42), Miss Alice Press (42–43); Social Secretary, Miss Gladys Fleishman (41–42), Miss Eileen Press (42–43); Corresponding Secretary, Mr. Julius Ozick (41–42), Miss Marian Lipschutz (42–43); Treasurer, Miss Helen Sigel (41–42), Mr. Stanley Braun (42–43); Editor *Math X*, Mr. Frank Grace (41–42), Mr. Harold Lewis (42–43); Associate Editor, Mr. Maxwell Levy (41–42), Mr. Hal Cooper (42–43); Head Coach, Mr. Levy (41–42), Mr. Lew Fite (42–43).

Mathematics Society, Brooklyn College

In the fall term of 1941 talks were presented on *Probability* and on *Geometry*. The titles were *The introduction to probability* by Leonard Greenstone, *The introduction to probability, continued*, by Bernice Schwartz, *Some paradoxes of probability* by Dr. James Singer, *A Euclidean finite geometry* by Mr. Bernard Greenspan, *Regular polygons, and regular and reguloid solids* by Professor H. F. MacNeish. The semi-annual integration contest of the club was conducted in December, with Julius Vogel as individual high scorer (16 out of 18 correct) and with a winning team from Professor Merle Bishop's class consisting of Leon Baker, Shirley Bloom, Bernice Hyman, Arthur Howick,

and Sylvia Teich. The annual Christmas party on December 23 concluded the program for the first semester. Officers were: President, Frank Bausch; Vice-President, Jay Weinstein; Secretary, Aida Kalish; Treasurer, Peter Chiarulli; Social director, Phyllis Cohen; Publicity Director, Arthur Zeichner. In the Spring term of 1942, lectures were given on *Mathematics in defense* by Professor H. F. MacNeish, *Vector fields* by Dr. James Singer, and *Graphical solution of the imaginary roots of the cubic equation* by Dr. Jack Wolfe. Officers for the Spring term were: President, Peter Chiarulli; Vice-President, Harvey Casson; Secretary, Frances Brand; Social Director, Bernice Schwartz; Publicity Director, Albert Blank.

Kappa Mu Epsilon, State Teachers College, Wayne, Nebraska

Following the organization in September and a *Kappa Mu Epsilon* party in October, seven program meetings were held once a month at which the following topics were presented: *Mathematics used in surveying* by Don Strahan, *Mathematical geography* by Earl Prousse, *Mathematics in defense* by Bob Dale, *History of the slide rule* by Richard Hedglin, *The dozen numbering system* by Margie Morgan (a particularly successful talk after which the audience worked problems using the new system), *Mathematics of gunnery* by LeRoy Thomsen, and *Mathematics as used by the Navy* by Ensign Jim Ahern. The last speaker is a past president of our Nebraska Alpha Chapter, just returned from his study at Annapolis, and his talk provoked plenty of questions from the boys. The program of the year closed in May with a joint banquet with *Lambda Delta Lambda*. Officers for the year were: President Leibnitz, Margie Morgan; Vice-President Archimedes, Don Strahan; Secretary Galileo, Homer Scace; Treasurer Einstein, Russell Vlaanderen; Reporter Gauss, Marjorie Johnson; Historian Pascal, Ruth Lundberg; Faculty Sponsor and Corresponding Secretary, E. Marie Hove.

Mathematics Club, Oberlin College

The program for the year included regular meetings with one or two speakers each, a Christmas party, and the annual banquet. Topics were presented as follows: *Transformations by paper folding* by Roselyn Siegel, *The Chicago meetings* by Professor Mary E. Sinclair, *Revolving numbers* by Dora Sherman, *Groups, a mathematical merry-go-round* by William Fishback, *Algebra of the ninth century* by Doris Miller, *Complex numbers* by Daniel Cowgill, *Spherical trigonometry and applications* by Bolton Strauch, *Taking a line apart* by Dr. R. W. Wagner, *Hyperbolic functions and applications* by Richard Hayden, *Triangles with sixty degree angles* by Arthur Oshlag, *Infinity* by Elizabeth Miller, *Quaternions* by Robert Kelner, *Finite geometries* by Allen Strehler, *Dimensional analysis* by Edgar Everhart, *Line coordinates* by Frederick Grannis, *Things that go up must come down* [ballistics] by Professor L. W. Taylor. Officers for the club were: President, Allen Strehler; Vice-President, Roselyn Siegel; Treasurer, Richard Hayden; Secretary, Margaret Sigler; Faculty Adviser, Professor Marie M. Johnson.

Junior Mathematics Club, The Iowa State College

The *Junior Mathematics Club* at the Iowa State College meets two or three times per quarter. Its aim is to promote interest in mathematics. A means to this end is finding interesting speakers (usually undergraduate) to discuss some mathematical topic at the freshman and sophomore level. Mailing cards and posters are used to announce the meetings, emphasizing a striking feature of each meeting. More than sixty students attended an informal draft party at the first meeting. They were assigned in three groups to different classrooms, and subjected in turn to three short mathematical tests entitled *K. P. duty*, *Maneuvers*, and *Medical department*. A colonial lighting arrangement of candles prevailed during the examinations. For faithful performance of duty each participant was granted an honorable discharge from the meeting, entitling the bearer to a fair share of refreshments. Prizes were awarded to the winners. The titles for the other meetings included *Magic squares*, *Primitive counting, abacus*, *Napier's bones*, *Statistical meeting*, *Ciphering match*, *Readings, the workers A-B-C*, *Musical meeting*, *Speedy computation*, and *Computation shortcuts*. The attendance varied between forty and eighty. The program committee consisted of the

following undergraduates: Misses Lucille Neff, Frances Wilson, Eleanor Hoeflin and Glee Barth, and the faculty adviser, Mr. Fred Robertson.

Junior Mathematical Club, University of Chicago

During the past year the following papers were presented by students: *Cardinal numbers* by Jack Indritz, *A test for goodness of fit* by Charles Stein, *The algebraic solution of equations* by W. C. Carter, *Some applications of classes to probabilities* by George Platzman, *Singularities on plane curves* by Alice Turner, *Linear diophantine equations* by Daniel Zelinsky. To Miss Turner was awarded a copy of Coxeter's *Non-Euclidean geometry* as a prize for the best student paper in content and presentation. In addition to the student papers, the following talks on pure and applied mathematics were presented to the club: *Nerve-fiber network in steady state activity* by A. S. Householder, (department of biophysics, University of Chicago), *Some mathematical aspects of cryptography* by Professor A. A. Albert, *Some applications of mathematics to psychology* by R. K. Meister (department of psychology, University of Chicago), *A mathematical analysis of extra-tropical storms* by F. L. Martin (Institute of Meteorology, University of Chicago), and *Some applications of groups* by Professor A. C. Lunn. In addition to the teas given before the talks, the Club sponsored two parties during the autumn and winter quarters, and a picnic in the spring quarter. The officers for the year 1941–42 were: President, Roy Dubisch; Social Chairman, Florence Dorfman; Treasurer, Jack Indritz; Committee, Albert Cahn, Anne Lewis, William Massey, Richard Schafer, Alice Turner, Ernest Wilkins, Jr., and Daniel Zelinsky. The officers elected for 1942–43 were: President, Anne Lewis; Social Chairman, Janet McDonald; Treasurer, Daniel Zelinsky; Committee, A.R. Jacoby, Hyman Zimmerberg.

The Cooper Union Mathematics Club

The program for the year included two moving pictures on mathematical topics and three other papers: *Rate of change*, a moving picture, *Number theory* by Harold Grad, *Calculus of variations* by S. Roth, instructor, *Forced vibrations*, a moving picture, *Calculus of finite differences* by J. M. Diamond. A book was awarded to W. Pepper, winner in the freshman contest in mathematics sponsored by the Club, and a *K. & E.* slide rule was awarded to Bernard Levine for excellence in first year mathematics. A team composed of the Club members Murray Klamkin, Harold Grad and Kenneth Robinson received honorable mention in the Putnam Mathematics Contest of 1942, and with the additional members D. R. Frankl and J. M. Diamond placed second in the Metropolitan Intercollegiate Mathematics Contest of 1942. Officers for the year were: President, Murray Klamkin; Vice-President, Kenneth Robinson; Secretary, Harold Grad; Treasurer, Milton Rabinowitz; Faculty Adviser, J. K. L. MacDonald.

Echols Mathematics Club, University of Virginia

Topics presented at the four meetings were: *Continued fractions* by Mr. Paul White, *Some properties of continued fractions* by Mr. Frank Myers, *An application of continued fractions* by Dr. G. A. Hedlund, *Some applications of circular inversion* by Mr. Walter Gottschalk. The officers for the year were: President, T. P. Botts; Vice-President, Mariano Garcia; Secretary-Treasurer, R. R. Bernard.

Pi Mu Epsilon, University of Missouri

At the first of eight monthly meetings a series of odd mathematical problems were presented to the twelve members present. At later meetings talks were given as follows: *The tautochrone* by Mr. Francis Abel, *Cryptography* by Dr. G. E. Schweigert, *The application of trigonometric series to X-ray analysis* by Mr. Paul Sharrah of the Physics Department, *The theory of dimensionability* by Dr. Ray Dufford of the Physics Department, *The theory of scientific concepts* by Professor Lewis Hahn of the Philosophy Department. The two May meetings were devoted to the election of new members and to the annual banquet, at which Professor W. D. A. Westfall was toastmaster. After the banquet the following officers were elected for the year 1942–43: Director, Paul Lerret; Vice-

Director, William Becker; Recording Secretary, Frances Jackson; Corresponding Secretary, Eugene Jackson; Treasurer, W. R. Utz.

Mathematics Club, Butler University

At three of the meetings the following papers were presented: *The fourth dimension* by Joseph Berry, *Mathematics and art* by Jane Gibson, *A survey of the background and trends in plane geometry textbooks* (thesis) by Maribelle Foster. Other programs included a report by Robert Stump on the part mathematics plays in our nation's war effort, a showing of a group of slides on the history of arithmetic (at two meetings), and the annual Christmas party which featured mathematical games and puzzles and a gift exchange. Officers for the year 1941–42 were: President, Maribelle Foster; Vice-President, Robert Stump; Secretary, Jane Gibson; Treasurer, Joseph Berry; Faculty Adviser, Mrs. Juna L. Beal.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Plane and Spherical Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1942. 9+275 pages. $2.00.

In writing this text the author has accomplished his objective. "The primary purpose of this book is to present in a sound pedagogical manner the usual course in trigonometry as offered in colleges and technical schools." The book is written in flexible units so that it can be adapted to different types of courses. But more important than this is the clear and careful exposition, well selected illustrative examples, and excellent lists of exercises.

The first quarter of the book contains material based on the trigonometric functions of acute angles. Geometrical, physical, and mechanical applications of right triangles are numerous. A chapter on approximate numbers and computation is fortunately included. The introduction of logarithms and their use in the solution of right triangles concludes this part.

The definitions of the trigonometric functions of a general angle are followed by the reduction formulas and solution of oblique triangles. The proofs of the laws are well chosen and check formulas are prominent. A course in computational trigonometry could end here.

Both the radian and the mil measurement of angles are given. Then the line values and graphs of the trigonometric functions are presented. The inverse functions are discussed fully as to principal values and illustrated graphically. Trigonometric formulas, identities, and both single and simultaneous equations are treated comprehensively. A chapter on complex numbers completes the analytic part of plane trigonometry.

The fifty pages devoted to spherical trigonometry start with definitions and statements of propositions from solid geometry. Napier's rules are proved and then used in the proof of the laws of sines and cosines. Right spherical triangles and six cases of oblique triangles are considered with the usual formulas for solution and check. Applications refer to the terrestrial and celestial spheres.

The printing and set-up is attractive and the diagrams are well drawn. Answers are included for odd numbered exercises. A protractor and four-place tables of trigonometric functions are provided while the text may be obtained with the complete Macmillan Logarithmic and Trigonometric Tables bound in the back.

MARIE M. JOHNSON

Military and Naval Maps and Grids. By W. W. Flexner and G. L. Walker, New York, The Dryden Press. 1942, 96 pages. $1.00.

The properties of five maps are discussed, namely, Gnomonic (or Great Circle), the Mercator, the Lambert Conformal Conic with Two Standard Parallels, the Stereographic, and the American Polyconic. No mathematical equipment for the reader beyond plane trigonometry is presupposed, and only basic mathematical properties of the maps are treated. Emphasis is upon the practical use of these maps for solving naval and military problems including aerial and terrestrial great circle navigation, determination of position by radio bearings, radio location of aircraft, military grid systems, and aerial photography. An extensive and well selected set of problems is included. Altogether this little book presents a brief and elementary organization of material that is interesting in itself and extremely valuable for the prosecution of the war.

R. E. GILMAN

A Review of Arithmetic. By Z. L. Smith. The Institute of Military Studies, University of Chicago, 1942, 37 pages planographed. 15¢.

The topics covered comprise addition, subtraction, multiplication, division, common fractions, decimals, ratio and proportion together with exercises and answers. The exposition is brief but clear. Anyone who has forgotten arithmetic will find this inexpensive little book a useful aid in recapturing at least the elements.

R. E. GILMAN

Calculus. By A. L. Nelson, K. W. Folley, and W. M. Borgman. Boston, D. C. Heath and Company, 1942. 10+356 pages. $2.75.

The authors have written a carefully planned and usable textbook for the beginning calculus. The outstanding feature of the text is the early introduction of integration. This is done by devoting the first three chapters to functions which are powers of x and polynomials. The differentiation and integration of these functions together with the usual applications to geometry and physics are thus given to the student before he needs to develop technique for more

complicated functions. Consequently, the student may concentrate in the early part of the course on principles and applications, and the calculus presents an immediate usefulness to him.

Definitions and theorems are carefully worded and displayed so that they may be found easily. The list of theorems on limits in the first chapter is fuller than usual and will help the student with some of his difficulties. No attempt is made to prove theorems which are beyond the capacity of the average beginning student, but the need for proof is pointed out and references for such proofs are given. The illustrations are good and many illustrative examples are worked in the text. There is an ample supply of exercises, which casual sampling show to be well chosen. Useful notations such as the use of capital letters to denote the principal values of the inverse trigonometric and the inverse hyperbolic functions and the use of the logarithm of the absolute value of a function aid in preventing the student from making errors in applications.

The text contains the usual subjects allotted to a first course in the calculus. The authors claim that the material may be covered in one year by a class that meets four times a week. The book varies perhaps in having chapters on curve tracing, hyperbolic functions, and differential equations. The chapter on series is quite full, the elementary theory of partial differentiation is given, and the chapter on differential equations gives the most common elementary forms of the ordinary differential equation together with their applications. The book closes with tables of integrals, natural logarithms, exponential and hyperbolic functions, and the trigonometric functions for radian measure.

MARIE J. WEISS

Basic Mathematics. By W. W. Hart. Boston, D. C. Heath and Company, 1942. 6+456 pages. $1.52.

In his preface the author introduces *Basic Mathematics* as a survey course in secondary mathematics for either classroom use or self-instruction. He suggests that the second half of the book is designed as a refresher course, but the reviewer is inclined to think that the entire book is useful for that purpose alone. The book is definitely condensed. It treats arithmetic, plane and solid geometry, algebra (through the binomial theorem), logarithms, and trigonometry of the right triangle all in the space of 450 pages. The condensation is, in general, well done. The wording is quite precise; important statements are italicized; and only material relevant to the main line of development is included. It is for these very reasons that the reviewer finds the book suitable only for review work. There is not sufficient drill on varied interpretations of the ideas introduced to form the basis of a thorough classroom course, and the style is too concise for the book to serve as a basis for self-instruction by a student who has not covered the material before. However, the book should prove valuable as a refresher course and reference book on secondary mathematics.

One interesting feature is the author's selection of exercises. He makes a definite attempt to show the application of the mathematics involved to prac-

tical problems which have come into prominence in connection with the war effort. Problems deal with the loaded weight of bombers rather than the number of eggs in a basket, with mensuration of airplane wings and machine parts rather than rooms and tables, with triangulation to obtain artillery ranges rather than to find the height of monuments. It is impossible to escape the impression that quite a number of these examples serve not so much to increase the value of the book as to popularize it by giving the impression that it is "up to the minute," but undoubtedly the book gains by the author's choice of exercises.

Several specific faults appear, the most glaring of which are as follows. Logarithms are introduced on page 177, and some 20 pages are devoted to their theory and applications. However, the laws of exponents are not discussed thoroughly until page 415. A second unfortunate point occurs on page 398. A recapitulation is given there in the form of an outline of the types of numbers. From this outline it appears that all real numbers are algebraic and that real numbers are not complex numbers.

The reviewer would recommend chapters VII and VIII on synthetic plane and solid geometry as the soundest portion of the book. The author proves congruence of triangles by superposition, but he is sufficiently broadminded to note that the congruence theorems can well be introduced as postulates and that this is frequently done. The section on the geometry of the sphere is to be commended particularly. Sufficient preparation in geometry is given there to enable the student to handle spherical trigonometry once he has mastered plane trigonometry. Unfortunately, the author cannot resist the temptation to discuss the circumference and area of circles without giving an adequate discussion of the notion of a limit.

M. E. MUNROE

General Trade Mathematics. By E. P. Van Leuven. New York, McGraw-Hill Book Co., Inc., 1942. 10+575 pages.

This book presents in a clear, concise fashion the arithmetic and mathematics required by the carpenter, machinist, and electrician. It is directed to students in technical high schools but is well adapted to home study by workmen with little training. The academic contents, consisting of arithmetic, elementary geometry, and simple equations, are applied to a wide variety of problems which include construction costs, gear speeds, screw threads, and indexing of milling machines. The last chapters, dealing with mechanics and electricity, are on the level of a high-school physics course.

Within the chapters, material is divided into easily understood unit topics, in which the order of presentation is: definitions, rule, examples, exercises, problems. Only at times is a derivation of the rule indicated. About half of the chapters are concluded by review problems. It is mentioned that the type of work determines the accuracy that may be expected, but the number of decimal places rather than the number of significant figures is given as a criterion. The

use of bold face type and boxing of examples contribute in making the book useful as a reference as well as a text.

E. L. CROW

Spherical Trigonometry with Naval and Military Applications. By L. M. Kells, W. F. Kerns, and J. R. Bland. New York and London, McGraw-Hill Book Company, Inc., 1942. 13+163 pages. $1.50. (With tables $2.40.)

This timely text for a pre-induction course in mathematics consists of a reprint of certain sections from the authors' *Plane and Spherical Trigonometry*, second edition (1940)* augmented by further applications to navigation and related topics. The extent of the text is indicated by the chapter headings: I. Logarithms; II. Review of plane trigonometry; III. The right spherical triangle; IV. Elementary applications; V. The oblique spherical triangle; VI. Applications; and the appendices: A. The mil; B. The range finder; C. Stereographic projections; D. Vectors and relative movement problems. Chapters I, III, V, VI and the first three appendices in the main coincide with subdivisions of the same name in the *Plane and Spherical Trigonometry* and have been adequately treated in reviews of that book.

In regard to the new material, Chapter III includes an additional section treating an oblique spherical triangle by solving two right spherical triangles. Chapter IV defines the navigational terms used and discusses the length of an arc on a parallel of latitude, plane sailing, middle latitude sailing, and the Mercator chart. Chapter VI has been augmented by a mention of the Ageton Method and the Dreisonstok method of solving oblique spherical triangles, a treatment of the Summer line of position method for making a fix, and an essay on aerial navigation. Appendix D applies vector addition to certain problems in the alignment and relative movement of a group of ships.

The text is well written. Students' difficulties are anticipated and warnings of possible pitfalls are given. An important feature is an emphasis on the form of computation. A more than ample collection of exercises of varying difficulty, many well illustrated, is included.

G. L. WALKER

Essentials of Astronomy. By J. C. Duncan, New York and London, Harper and Brothers, 1942. 179 pages. $1.85.

The ideal way to teach astronomy would be to have each student observe and record the daily, monthly, seasonal and yearly phenomena from early childhood to his college years. If such a student could also have the advantage of making his observations at several places, from a far northern to a far southern latitude, so much the better.

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* Reviewed in this MONTHLY, vol. 47, 1940, pp. 703–704 by J. M. Feld. The first edition of the same book was also reviewed in this MONTHLY, vol. 43, 1936, p. 39 by J. M. Feld.

A college teacher who had such a class with such a background, would have an ideal opportunity to develop a course which should give a clear and full appreciation of the scientific method, as applied to the observational data at the disposal of his students.

In the brief book before us Professor Duncan has provided such a background of observational material in chapter one. In the second chapter the observational material is interpreted.

So skillfully woven into this material are the definitions and principles that the student is led to their use as the natural and easy way of expressing the new ideas which unfold in his mind.

Chapter three gives a brief account of the discovery of the laws of motion and the law of gravitation and the importance of these laws in explaining all of the phenomena of motion in the solar system.

Chapter four deals with radiation and introduces the student to some of the instruments which enable us to interpret the messages which the radiations from the heavenly bodies bring. The elementary principles of the telescope and the spectroscope are especially well presented.

In the brief space allotted to discussing gravitation and radiation in chapters three and four, the author gives a fine presentation of the part they play in producing the phenomena we observe in the universe.

The remaining three chapters give a brief account of the various celestial bodies which modern instruments have brought within our field of observation. The book concludes with a presentation of our present conception of the structure of the universe.

The appendices contain fourteen tables of useful data and information, followed by the very excellent star maps, which have been such a valuable feature of Professor Duncan's "Astronomy."

The publishers are to be commended for their part in producing this book, with its fine reproductions of astronomical photographs and diagrams. The few typographical errors which it contains will undoubtedly be corrected in a second reprint.

The relative brevity of the section dealing with astrophysical subjects, together with the excellent star maps, should recommend the book as a desirable text for courses in pre-aviation astronomy.

The writer was charmed with this book when it was first read, and the charm was further enhanced after using it as a text in his introductory course in Astronomy given in the summer session at Cornell during July and August 1942.

S. L. BOOTHROYD

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 561. *Proposed by Howard Eves, Syracuse University*

Given two triangles inscribed in the same circle and such that the Simson lines with respect to one triangle of the vertices of the other are concurrent (as in E 535), prove that the Simson lines with respect to the two triangles of a point on the common circumcircle are parallel.

E 562. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of the form $ab0cd$ whose square contains the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

E 563. *Proposed by N. A. Court, University of Oklahoma*

Let A', B', C', D' be the antipodes of the circumcenter O of a tetrahedron $ABCD$ on the respective spheres $OBCD$, $OCDA$, $ODAB$, $OABC$. Show that the lines AA', BB', CC', DD' are generators of a quadric. May this quadric be a cone?

E 564. *Proposed by Ivan Niven, Purdue University*

Let a, b, and n be any positive integers such that n divides a^n-b^n. Prove that n divides $(a^n-b^n)/(a-b)$.

E 565. *Proposed by H. W. Becker, Mare Island Submarine Base*

Show that the number of ways n men can be divided up in crews is N_n, in the notation of E 461 [1941, 701]. What is the number of ways if two particular men cannot stand the sight of each other, and must be kept in different crews? What is it if one must be segregated from each of m others?

SOLUTIONS

Exponential Limits

E 528 [1942, 404]. *Proposed by R. A. Rosenbaum, Reed College*

Prove and generalize the identity

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k}=e^{-1/2}.$$

I. *Solution by A. M. Peiser, Cornell University.* This is easily proved by noting that

$$\log\left\{\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k}\right\}=-k+k^2\left\{\frac{1}{k}-\frac{1}{2k^2}+o\left(\frac{1}{k^2}\right)\right\}$$
$$=-\tfrac{1}{2}+o(1)$$

as $k\to\infty$.

One immediate generalization is

$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^{k^2}\cdot e^{-kx}=e^{-x^2/2}.$$

II. *Solution by Alexander Beckerman, New York City.* The generalization

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^n}\exp\left(\sum_{r=1}^{n-1}\frac{(-1)^r k^{n-r}}{r}\right)=\exp\left(\frac{(-1)^{n-1}}{n}\right)$$

may be proved by noting that

$$k^n\log\left(1+\frac{1}{k}\right)+\sum_{r=1}^{n-1}\frac{(-1)^r k^{n-r}}{r}=k^n\sum_{r=n}^{\infty}\frac{(-1)^{r-1}k^{-r}}{r}$$
$$=\frac{(-1)^{n-1}}{n}+o(1)$$

as $k\to\infty$.

Also solved by B. H. Bissinger, L. M. Kelly, H. D. Larsen, Melvin Lax, J. Rosenbaum, and P. D. Thomas. Lax combined the two generalizations, proving that

$$\lim_{k\to\infty}\left(1+\frac{z}{k}\right)^{k^n}\exp\left(\sum_{r=1}^{n}\frac{(-1)^r}{r}k^{n-r}z^r\right)=1.$$

An In- and Circum-scriptible Hexagon

E 529 [1942, 404]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Construct an irregular hexagon which shall be both inscriptible and circumscriptible.

Solution by the Proposer. CONSTRUCTION. On a circle take two diametrically opposite points A and D, and another point B, with the restriction that AB be unequal to the radius. Then locate a fourth point C, on the semicircle ABD, such that the sum of AB and DC is equal to the diameter. Finally, locate E and F, the images of C and B by reflection in AD. The hexagon $ABCDEF$ will fulfil the conditions of the problem.

Proof. The essential feature is that the bisectors of the angles ABC and BCD meet on AD, so that (on account of the symmetry of the figure) all the angle bisectors are concurrent, and the hexagon has an incircle.

NOTE. After this special (symmetrical) hexagon has been constructed, and its circumcircle and incircle drawn, then by means of Poncelet's Porism the most general such hexagon can be constructed. [See this MONTHLY, 1942, 364.]

The Six Radical Spheres

E 530 [1942, 404]. *Proposed by P. D. Thomas, Southeastern State College, Okla.*

Is there a sphere orthogonal to the six radical spheres determined by four given spheres whose centers are not coplanar? (The radical sphere of two spheres is the locus of a point whose two powers have zero sum.)

Solution by the Proposer. The answer is Yes. The intersection of the six radical planes of the four given spheres is their radical center, which, having equal powers with respect to the four spheres, is the center of a sphere, S, orthogonal to them. But the radical sphere of any two of the four given spheres is coaxal with those two. Also a sphere orthogonal to two spheres is orthogonal to every sphere of the coaxal pencil determined by them. Thus the sphere S is orthogonal to the six radical spheres as well as to the four original spheres. (See N. A. Court, *Modern Pure Solid Geometry*, 1935, pp. 179, 186, 201, 202.)

The Meigs Hall Problem

E 531 [1942, 475]. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle?

I. *Solution by R. K. Allen, Montpelier, Vermont.* The students may finish the examination in 6! or 720 ways. In order that they may finish so that no student has to pass over any other student the first one to finish must be one of the two on the ends, which may be done in either of two ways. The second to finish must be one of the two then sitting on the ends, and so on. There are 2^5 or 32 ways in which no student will have to pass over some other student. The chance that some student will have to pass over some other student is accordingly

$$(720 - 32)/720 = 43/45.$$

II. *Solution by Howard Eves, Syracuse University.* Given n students, the probability that the first man finished will not have to pass over any of the others in gaining the aisle is obviously $2/n$. Hence the required probability for our problem is

$$1 - \frac{2}{6}\,\frac{2}{5}\,\frac{2}{4}\,\frac{2}{3} = \frac{43}{45}.$$

Also solved by D. F. Barrow, H. W. Becker, and W. E. Buker.

Five Consecutive Digits Forming a Square

E 532 [1942, 475]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a perfect square whose digits form one of the permutations of five consecutive digits.

Solution by W. E. Buker, Pittsburgh Public Schools. If N is a perfect square, then $N \equiv 0, 1, 4$, or $7 \pmod 9$. So N must be one of the permutations of either

01234 or 34567.

Of the 120 permutations of 01234, we omit those whose first digit is 0, and those whose last digit is 0, 2, or 3. Of the 36 remaining possibilities, we note that all squares ending in 1 or 4 have an even number in the tens place. This eliminates twelve more, leaving 24 numbers to look up in a table. We find that

23104 and 32041

are the squares of 152 and 179.

Of the permutations of 34567, we note that squares do not end with 3 or 7, nor with 5 unless preceded by 2. Thus we have 48 possibilities. But squares ending in 4 have an even number in the tens place, while those ending in 6 have an odd number in the tens place. There are again 24 numbers to look up, and it is found that none of these are squares.

Also solved by R. K. Allen, Howard Eves, R. V. Heath, Aida Kalish, and the proposer. Allen remarks that the only values of n less than 1000 such that the digits of n^2 are a permutation of consecutive integers are 18, 24, 66, 74, 152, and 179.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4075. *Proposed by N. A. Court, University of Oklahoma*

If the radical center of four spheres coincides with the Monge point of the tetrahedron (T) determined by their centers, the tetrahedron (S) formed by the four radical planes of the given spheres with their orthogonal sphere is orthological to the twin tetrahedron (T') of (T) (*i.e.*, the perpendiculars dropped from the vertices of (S) upon the corresponding faces of (T') are concurrent).

4076. *Proposed by Harold E. Gove, Major, Washington, D. C.*

Given the triangle $A_1A_2A_3$ show how to construct the triangle $B_1B_2B_3$ so that the triangles $B_iB_iA_k$ will be equilateral and exterior to $B_1B_2B_3$.

4077. *Proposed by V. Thébault, San Sebastián, Spain*

If four spheres, each passing through a corresponding vertex of a tetrahedron $ABCD$, intersect in pairs on the corresponding edge, the four spheres are concurrent in a point M (S. Roberts). Show that: (1) The points A', B', C', D' diametrically opposite to A, B, C, D on the corresponding spheres are in a plane (P) passing through M (R. Bouvaist). (2) The plane (P) is a Simson plane of the tetrahedron $A_1B_1C_1D_1$ formed by the planes parallel to the planes of BCD, CDA, DAB, ABC and passing respectively through A', B', C', D'.

SOLUTIONS

Twin Tetrahedrons

4024 [1942, 128]. *Proposed by N. A. Court, University of Oklahoma*

Given two twin tetrahedrons $(T)\equiv ABCD$, $(T')\equiv A'B'C'D'$ (see the proposer's *Modern Pure Solid Geometry*, p. 58, art. 191), consider the tetrahedron (A') formed by the face BCD of (T) and the three planes forming the trihedral angle A' of (T'); let (B'), (C'), (D') be the analogous tetrahedrons for the vertices B', C', D' of (T'). Show that the twelve-point spheres of the tetrahedrons (A'), (B'), (C'), (D') (*ibid.*, p. 251, art. 764) are tangent to the twelve-point sphere of (T).

Solution by the Proposer. The diagonal AA' of the common circumscribed parallelepiped of (T) and (T') is trisected internally by the face BCD of (T), say, in L, and the point L is the centroid of the triangle BCD.

On the other hand, the edges of the trihedral angle A' of (T') pass through the mid-points of the sides of the triangle BCD. Consequently, the two tetrahedrons (A'), (T) correspond to each other in the homothecy $(L, -2)$ having L for homothetic center and -2 for homothetic ratio.

Now the twelve-point spheres of the two tetrahedrons (A'), (T) correspond to each other in the homothecy $(L, -2)$, and both spheres pass through the point L, hence they touch each other at this point. Similarly for the tetrahedrons (B'), (C'), (D').

Note. The above proposition is an extension to space of the following proposition in the plane.

The parallels to the sides of a triangle ABC through the respectively opposite vertices form a triangle $A'B'C'$. Show that the nine-point circles of the triangles $A'BC$, $B'CA$, $C'AB$ are tangent to the nine-point circle of the triangle ABC (*Nouvelles Annales de Mathématiques*, 1865, p. 322, Q. 722).

THE MATHEMATICAL ASSOCIATION OF AMERICA

Because of the cancellation of the New York meeting of the Mathematical Association of America at the request of the Office of Defense Transportation, the business of the Board of Governors for 1942 was completed by mail vote. The outgoing Board voted

1. To elect B. W. Jones Associate Secretary for a term of five years.
2. To adopt the following motion:

Members who are in active war service may, on their request, be exempt from dues, not receiving the MONTHLY but otherwise being members in full standing. This is particularly appropriate when they are not able to receive the MONTHLY or the official announcements.

3. To transfer $2,000 in Defense Bonds to General Endowment, this to be grouped with "Investments."
4. To adopt the following motion jointly with the American Mathematical Society Council:

That a War Policy Committee of the A.M.S. and the M.A.A. be appointed jointly by the incoming presidents, and a sub-committee to handle at once the problem of allocation of teachers, the work of this sub-committee to be absorbed with that of the War Policy Committee; and that President-elect Stone act as representative of the two organizations in Washington until such time as the War Policy Committee will function.

5. To elect to membership the following thirty-eight persons on applications duly certified:

P. R. ANNEAR, M.S. (Case) Asst. Prof., Acting Head of Dept., Math. and Astr., Baldwin-Wallace Coll., Berea, Ohio

A. V. BAEZ, A.M. (Syracuse) Asst. Prof., Math. and Physics, Wagner Coll., Staten Island, N. Y.

C. W. BAILEY, Reporter, Cleveland Press, Cleveland, Ohio

HELEN P. BEARD, A.M. (Pennsylvania) Instr., Sophie Newcomb Coll., New Orleans, La.

F. P. BEER, Ph.D. (Geneva, Switzerland) Instr., Univ. of Kansas City, Kansas City, Mo.

R. L. BEINERT, A.B. (Hobart) Part-time Instr., Cornell Univ., Ithaca, N. Y.

S. G. BOURNE, B.S. (Rutgers) Jr. Instr., Johns Hopkins Univ., Baltimore, Md.

HERMAN BRANSON, Ph.D. (Cincinnati) Asst. Prof., Physics, Howard Univ., Washington, D. C.

R. C. BUCK, A.M. (Cincinnati) Jr. Prize Fellow, Harvard Soc. of Fellows, Cambridge, Mass.

W. B. CATON, Ph.D. (Yale) Instr., Illinois Inst. of Tech., Chicago, Ill.

LOUISE H. CHIN. Student, Univ. of California, San Francisco, Calif.

D. E. CHRISTIE, Ph.D. (Princeton) Instr., Physics, Bowdoin Coll., Brunswick, Me.

E. L. CROW, Ph.D. (Wisconsin) Math. Consultant, Research and Development Div., Bur. of Ordnance, Navy Dept., Washington, D. C.

BENJAMIN EPSTEIN, Ph.D. (Illinois) Physicist, Frankford Arsenal, Philadelphia, Pa.

H. W. EVES, A.M. (Harvard) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

J. W. GIVENS, JR., Ph.D. (Princeton) Asst. Prof., Northwestern Univ., Evanston, Ill.

V. H. HAAG, A.M. (Duke Univ.) Instr., Hershey Jr. Coll., Hershey, Pa.

R. W. HAMMING, Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill.

CECIL HASTINGS, JR., B.S. (Florida) Computer, Franklin Inst., New York, N. Y.

B. K. HOVEY, Ph.D. (Göttingen) Instr., Elec. Engineering, Univ. of Pittsburgh, Pittsburgh, Pa.

S. W. HOWELL, A.M. (South Dakota) Instr., Math. and Physics, Univ. of South Dakota, Vermillion, S. D.

MRS. LOUISE S. HUNTER, Ed.M. (Harvard) Asst. Prof., Virginia State Coll., Ettrick, Va.

IRVING KAPLANSKY, Ph.D. (Harvard) Instr., Harvard Univ., Cambridge, Mass.

V. O. MCBRIEN, Ph.D. (Catholic Univ.) Mathematician, U. S. Government, Washington, D. C.

LEONARD MCFADDEN, Ph.D. (Brown Univ.) Instr., Virginia Poly. Inst., Blacksburg, Va.

INGO MADDAUS, JR., Ph.D. (Michigan) Asst. Prof., Univ. of Oregon, Eugene, Ore.

N. A. PEEBLES. Statistical Clerk, Office of President, Atlantic Coast Line RR Co., Wilmington, N. C.

W. G. POLLARD, Ph.D. (Rice Inst.) Asso. Prof., Physics, Univ. of Tennessee, Knoxville, Tenn.

MRS. ANNIE N. ROWLAND, M.S. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Texas

REV. B. M. RUSSELL, A.B. (St. Viator's) Head of Dept., Fournier Inst., Lemont, Ill.

W. G. SCOBERT, A.B. (U.C.L.A.) Grad. Asst., Univ. of Oregon, Eugene, Ore.

MAURICE SINGER, B.S. (Tulane) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.

H. W. STEINHAUS, Ph.D. (Göttingen) Chief, Research Div., Group Dept., Equitable Life Assur. Soc., New York, N. Y.

E. C. STEWART, B.A. (British Columbia) Teacher, Math. and Science, Wells Barkerville School, Wells, B. C., Canada

H. I. TREIBER, A.B. (Brooklyn Coll.) Signal Corps Inspector, Newark Signal Corps Inspection Zone. *Res.* Brooklyn, N. Y.

Mrs. LOIS K. TURNER, M.S. (Virginia Poly. Inst.) Instr., University High School, Univ. of Minnesota, Minneapolis, Minn.

W. R. UTZ, JR., A.M. (Missouri) Instr., Univ. of Missouri, Columbia, Mo.

J. W. WILEY, A.M. (Michigan) Head of Dept., Math. and Physics, Anderson Coll., Anderson, Ind.

W. D. CAIRNS, *Retiring Secretary-Treasurer*

The election of officers for 1943 by the membership of the Association and by the Board of Governors was conducted by means of mail ballots. The results of the elections were as follows:

President for a two-year term: W. D. Cairns, Oberlin College.

Second Vice-President for a two-year term: C. C. MacDuffee, Hunter College.

Governors at large for three-year terms: Saunders MacLane, Harvard University, and E. J. Moulton, Northwestern University.

On the recommendation of the Editor-in-Chief, L. R. Ford, the following associate editors of the MONTHLY have been elected for the year 1943:

L. M. Blumenthal
N. B. Conkwright
H. S. M. Coxeter
W. M. Davis
Otto Dunkel
B. F. Finkel
J. S. Frame
Orrin Frink, Jr.
Marjorie J. Groves
M. R. Hestenes
B. W. Jones
R. E. Langer
J. R. Musselman
C. V. Newsom
Virgil Snyder
Marie J. Weiss

W. B. CARVER, *Secretary-Treasurer*

FINAL REPORT OF THE SECRETARY-TREASURER AS TREASURER, JANUARY 15, 1943

RECEIPTS

Balance Dec. 20, 1941		$6,324.98
1941 indiv. dues	$ 347.65	
1941 instit. dues	21.00	
1941 subscriptions	26.08	
1942 indiv. dues	6,633.96	
1942 instit. dues	681.10	
1942 subscriptions	819.69	
Initiation fees	224.00	
Advertising	609.36	
Reprints	219.64	
Sale copies of MONTHLY	158.51	
Sale First Carus Mon.	12.50	
Sale Second Carus Mon.	12.50	
Sale Third Carus Mon.	20.00	
Sale Fourth Carus Mon.	8.75	
Sale Fifth Carus Mon.	47.18	
Sale Sixth Carus Mon.	474.88	
Sale Archibald's *Outline of Hist. of Math.*	123.71	
Sale library books and periodicals	575.16	
Sale Rhind Papyrus	104.60	
Annals subscriptions	7.50	
Duke Journal subscriptions	6.00	
Math. Reviews subscriptions	13.00	
Drury Coll. int. Hardy Fund	120.00	
Refund office expense	7.21	
Refund War Prep. reprints	13.84	
Repaid from Carus Fund	595.60	
Interest	1,189.80	
Payment from restricted Carus Fund	49.70	
Payment from restricted Chace Fund	2.20	
Transfer certificate of deposit	1,120.77	
Restricted funds transferred from Carus Fund	298.20	
Restricted funds transferred from Chace Fund	13.20	14,557.29
Total 1942 receipts to date		$20,882.27

EXPENDITURES

Publisher's bills (Nov. '41–Oct. '42)		$ 5,955.09
Reprints		347.28
President's office		4.48
Editor-in-chief's office 1941		200.00
Editor-in-chief's office 1942		562.41
Expense Exec. and Finance Committees		233.85
Membership maintenance		150.00
War Preparedness Committee		39.96
Secretary-Treasurer's office		
Postage	$ 475.92	
Bond	11.26	
Office expense	155.35	
Express, tel., etc.	103.07	
Clerical work	2,863.08	
Printing	152.01	
Bank charge	28.19	3,788.88
Annals subvention		200.00
Duke Journal subvention		150.00
Math. Reviews subvention		500.00
Expense of sections		182.78
Expense acct. regional governors		169.84
Lehigh meeting		25.15
Poughkeepsie meeting		75.00
New York meeting and annual elec.		111.03
Paid *Annals* subscriptions		10.00
Forwarded *Annals* subscriptions		7.50
Forwarded *Duke Journal* subscriptions		6.00
Forwarded *Math. Reviews* subscriptions		13.00
Sust. memb. in Amer. Math. Society		100.00
Insurance back copies MONTHLY		4.55
Storage back copies MONTHLY		30.00
Paid back copies MONTHLY		177.50
Paid B. F. Finkel int. Hardy Fund		120.00
Refund subscriptions		5.85
Library expense		39.08
Moving Association office		193.37
Expense Ithaca office		44.95
War Policy Committee		113.44
Expense acct. Carus Committee		5.00
Chauvenet Prize Award		100.00
Balance purchase War Bond		200.09
Balance purchase Phelps Dodge Bond and accr. int.		26.15

RECEIPTS (continued)		EXPENDITURES (continued)	
		Transfer Peoples Bank savgs. acct. to Gen. Endowment	1,866.47
		Transfer Defense Bonds to Gen. Endowment	2,000.00
		From certificate of deposit for re-investment	715.09
Total expenditures	18,473.79	Total expenditures	$18,473.79
Balance to end of 1942 business	$ 2,408.48	Checking account	1,172.47
Received on 1943 business	1,288.85	Oberlin Savgs. Bk. acct. restricted*	811.20
		Cleveland Trust Co. savgs. acct.	1,713.66
Book balance Jan. 15, 1943	$ 3,697.33	Bank balance Jan. 15, 1943	$ 3,697.33

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CARUS MONOGRAPH FUND

Balance December 20, 1941		$6,392.82
Receipts: Sales	$575.81	
Interest	180.78	756.59
		$7,149.41
Expense acct. Carus Committee		5.00
Balance January 15, 1943		$7,144.41

ARNOLD BUFFUM CHACE FUND

Balance December 20, 1941		$8,163.98
Receipts: Sale Papyrus	$104.60	
Interest	230.46	335.06
Balance January 15, 1943		$8,499.04

JACOB HOUCK MEMORIAL FUND

Balance December 20, 1941	$8,181.17
Interest	224.81
Balance January 15, 1943	$8,405.98

CHAUVENET PRIZE FUND

Balance December 20, 1941	$ 632.94
Interest	14.40
	$ 647.34
Paid on Prize Award December 1941	32.94
Balance January 15, 1943	$ 614.40

LIFE MEMBERSHIP FUND

Liability on earlier life memberships as of January 1, 1942	$ 900.17
To be transferred to current funds, surplus	75.87
Liability on life memberships as of January 1, 1943	$ 824.30

* This is the balance, not yet distributed, of $2,704.00 which was tied up at the time of bank closures in 1933. It is probable that some of the balance, but not all, will be repaid.

PERMANENT INVESTMENTS OF THE ASSOCIATION

	Par Value	Market Value Dec. 31, 1942
U. S. Savings Bonds	$ 1,275.00	. 1,468.00
U. S. Treasury 1% Notes Ser. A 1946	3,000.00	2,979.00
U. S. Treasury 2¾% Bond 1947	1,000.00	1,042.50
U. S. Government 1¼% Bonds 1948	2,000.00	2,012.00
U. S. Treasury 2% Bonds 1950	3,000.00	3,051.00
HOLC 3% Bonds Ser. A 1944–52	3,000.00	3,096.00
Phelps Dodge Corp. 3½% conv. deb. 1952	1,000.00	1,047.50
U. S. Savings 2½% Bonds Ser. G 1953	3,000.00	3,000.00
U. S. Savings 2½% Bonds Ser. G 1954	8,200.00	8,200.00
Texas Power & Light Co. 5% First Mort. Bond 1956	1,000.00	1,075.00
Amer. Tel. & Tel. Co. 3% Bonds conv. 1956	2,000.00	2,140.00
Commonwealth Edison Co. 3½% Bonds. conv. deb. 1958	2,000.00	2,180.00
N. Y. Steam Corp. 3½% First Mort. Bond 1963	1,000.00	1,061.25
Montana Power Co. 3¾% First Ref. Mort. Bonds 1966	3,000.00	3,112.50
Gatineau Power Co. 3¾% First Mort. Bond Ser. A 1969	1,000.00	918.75
Penn. R. R. Co. 3¾% Genl. Mort. Bonds Ser. C 1970	2,000.00	1,775.00
Cols. & So. Ohio Elec. Co. 3¼% First Mort. Bonds 1970	2,000.00	2,160.00
Shawinigan W. & P. Co. 4¼% First Mort. Bonds 1970	2,000.00	2,010.00
C. & O. Ry. Co. 3½% Ref. Mort. Bonds Ser. D 1996	3,000.00	3,060.00
Land Trust Certif., Hotel Cleveland Site	700.00	510.00
	$45,175.00	$45,898.50

Of the funds on hand indicated in the first division of this report, $1,279.32 belongs to the Carus Monograph Fund, $574.04 to the Chace Fund, $224.81 to the Houck Fund, $114.40 to the Chauvenet Fund, while $824.30 is held as a Life Membership Fund, representing the liability as of January 1, 1943, on life memberships paid for earlier.

When the accounts were closed by the retiring secretary-treasurer, there remained on the total business for 1942 the following items:

BILLS RECEIVABLE		BILLS PAYABLE	
1942 individual dues	$200.00	Publisher's bills Nov., Dec. '42)	$1,250.00
Advertising	70.00	Subsidy *Duke Journal*	50.00
Reprints	35.00	Carus Monograph Fund	1,279.32
		Chace Fund	574.04
	$305.00	Houck Fund	224.81
		Chauvenet Fund	114.40
		Life membership fund	824.30
			$4,316.87

These items are about as usual and the current income of the Association amply provides for their payment as they come due.

The retiring secretary-treasurer holds receipts from the incoming secretary-treasurer for the assets in the general treasury and from the Cleveland Trust Company for the securities listed in "Investments."

W. D. CAIRNS, *Retiring Secretary-Treasurer*

THE SEVENTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The seventeenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 28, 1942. Professor C. O. Oakley, chairman of the Section, presided at the morning and afternoon sessions.

The attendance was about thirty-five, including the following twenty-six members of the Association: E. F. Allen, H. W. Brinkmann, P. A. Caris, Mary L. Constable, H. B. Curry, J. E. Davis, Arnold Dresden, V. V. Latshaw, Marguerite Lehr, F. L. Manning, A. E. Meder, Jr., Richard Morris, W. R. Murray, C. A. Nelson, C. O. Oakley, A. E. Pitcher, G. E. Raynor, C. J. Rees, I. J. Schoenberg, J. A. Shohat, L. L. Smail, G. L. Walker, A. D. Wallace, R. M. Walter, Anna Pell Wheeler, P. M. Whitman.

At the business meeting the following officers were elected for next year: Chairman, J. E. Davis, Drexel Institute of Technology; Secretary, P. M. Whitman, University of Pennsylvania. It was voted to hold the next meeting at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 27, 1943. The Section voted to cooperate in the formation of a local group of technical societies comparable to one existing in Chicago.

The following papers were presented:

1. "On a theorem of Jensen" by Professor I. J. Schoenberg, University of Pennsylvania.

2. "Exterior ballistics" by Professor G. E. Raynor, Lehigh University.

3. "On modern methods in the numerical solution of linear problems" by Dr. Hilda Geiringer, Bryn Mawr College, introduced by Professor Brinkmann.

4. "The Heaviside operational calculus" by Dr. H. B. Curry, Frankford Arsenal.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The basis of Prof. Schoenberg's discussion was Jensen's theorem: If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles. The first published proof of this theorem is due to Walsh, Annals of Mathematics, 22 (1920) pp. 128–144. This theorem is now extended to a set of points in space, furnishing information about the possible location of points of equilibrium of a particle attracted by the given points according to the inverse distance law of attraction. For the special case of a cubic polynomial $f(z)$ of given roots, a ruler and compasses construction of the roots of $f'(z)$ is given which remains valid if all the roots of $f(z)$ are real (see Walsh, *loc. cit.*, pp. 142–144).

2. In the first part of the paper Professor Raynor gave a description of the force system acting on a spinning projectile. Then a set of simplified equations of motion was discussed and it was explained briefly how ballistic and firing tables are prepared.

3. In order to obtain the effective solution of a system of n nonhomogeneous linear equations $a_{ik}x_k+r_i=0$ with determinant different from zero, Dr. Geiringer outlined a general iteration scheme which contained all particular cases so far known. It reduced respectively to: a) "iteration by simultaneous displacements" and b) "iteration by successive displacements." Well known particular cases of the first procedure are the "ordinary" iteration or iterated transformation of the error-vector $z_i^{(r)}=x_i^{(r)}-x_i$ (x_i=solution), and the "method of steepest descent." In the simplest case of b) one computes successively values $x_i^{(r+1)}$ of the ith unknown by means of the ith equation where the values of the other unknowns are $x_1^{(r+1)}, x_2^{(r+1)}, \cdots, x_{i-1}^{(r+1)}, x_{i+1}^{(r)}, \cdots, x_n^{(r)}$ ($r=0, 1, 2, \cdots$; $x_i^{(0)}$ arbitrary). It is known that this method converges if the matrix (a_{ik}) is positive definite; also in this case any order in iterating the equations is admissible. The procedure is identical with Southwell's Relaxation Method. Besides, necessary and sufficient conditions for the convergence of b) can be indicated, and new simple sufficient conditions are deduced none of which demands symmetry of the matrix. On the other hand, if the matrix is symmetric and $a_{ii}>0$ then positive definiteness is not only sufficient but also necessary for convergence from an arbitrary starting point. (The latter is true for the "ordinary" iteration but there the condition is not sufficient.) In each of these cases the convergence may be accelerated by "group-iteration."

4. Dr. Curry presented a rigorous theory of the Heaviside operational calculus which was algebraic in character and made no use of infinite integrals. Naturally it included only the discrete aspects of the calculus, *i.e.* applications to ordinary differential equations.

P. M. WHITMAN, *Secretary*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Associate Professor E. F. Beckenbach of the University of Michigan has been appointed to an associate professorship at the University of Texas.

Associate Professor C. S. Carlson of St. Olaf College has been promoted to a full professorship.

E. W. Holt of Lawrence Academy, Groton, Massachusetts is now a lieutenant, U.S.N.R., and is teaching in the Naval Academy Preparatory School, Naval Operating Base at Norfolk, Virginia.

Associate Professor E. D. Jenkins is on leave from Eastern Kentucky State Teachers College and is serving with the Navy Department at Corpus Christi, Texas.

Sidney Kaplan has been appointed assistant economic statistician of the Industry and Facilities Branch of the War Production Board and is stationed at New York.

Dr. R. H. Moorman of Tennessee Polytechnic Institute has been promoted to an assistant professorship.

J. K. Reckzeh of West Virginia Institute of Technology is now an Ensign teaching Navigation at the Navy Ground School, Corpus Christi, Texas.

Dr. Arthur Rosenthal is now an assistant professor at the University of New Mexico. This is in correction of an item in the November issue of this MONTHLY.

Associate Professor J. B. Rosser and Assistant Professor R. J. Walker have been granted leaves of absence from Cornell University. The former is serving with the National Defense Research Committee in Washington and the latter as a mathematician at Aberdeen Proving Ground.

Dr. Andrew Sobczyk is on leave from Oregon State College and is located at the Radiation Laboratory of Massachusetts Institute of Technology.

Dr. Ellen C. Stokes, instructor in mathematics at the New York State College for Teachers at Albany, has been appointed dean of women.

Professor W. L. Williams has been appointed head of the department of mathematics at the University of South Carolina to succeed Professor J. B. Coleman who recently resigned.

Professor Roscoe Woods has been serving as the acting head of the mathematics department of the University of Iowa since last September. Professor H. L. Rietz, the former head of the department, has relinquished the position on account of ill health.

The following appointments to instructorships are announced:
University of South Carolina: Miss Flora Dinkines
University of Southern California: E. C. Rex, Veryl Throckmorton and Miss F. Marian Clarke (visiting instructors)

Associate Professor Emeritus C. H. Currier of Brown University died on January 5, 1943. He was a charter member of the Mathematical Association.

Professor E. L. Dodd of the University of Texas died on January 9, 1943. He was a charter member of the Mathematical Association.

Professor Emeritus J. H. Scarborough of Central Missouri State Teachers College died on November 25, 1942. He was a charter member of the Mathematical Association.

MEXICAN CONGRESS OF MATHEMATICS

The First National Congress of Mathematics was held in Saltillo, State of Coahuila, Mexico, November 1–7, 1942, in connection with the celebration of the seventy-fifth anniversary of the founding of the Ateneo Fuente, a college located in Saltillo. The Congress was in commemoration of the three hundredth anniversary of the birth of Sir Isaac Newton.

The officers were: President, Dr. Alfonso Nápoles Gándara, head of the Department of Mathematics in the National University and of the Institute of Mathematics, recently organized to stimulate research; Vice-President, Dr. Carlos Graef Fernández, head of the Department of Astrophysics in the National University; Secretary, Ing. Francisco José Alvarez. Officers of the Section of Pure Mathematics were: President, Dr. Carlos Graef Fernández; Secretary, Professor Francisco Zubieta Russi; Reporter, Professor Alberto Barajas Celis. Officers of the Section of Applied Mathematics were: President, Dr. Manuel Sandoval Vallarta; Vice-President, Professor Leopoldo Medina; Reporter, Dr. Nabor Carrillo.

Special guests were Dr. Blas Cabrera, former director of the Institute of Physics of Madrid, Dr. Manuel Sandoval Vallarta of the Massachusetts Institute of Technology, and Dr. P. R. Rider of Washington University, exchange professor at the National University.

One hundred ten persons were registered, but very many more than this were in actual attendance at the sessions. Among the speakers at the general sessions were the Governor of the State of Coahuila, the Secretary of Public Education, the Rector of the National University, and a representative of the United States Ambassador. There were some twelve papers presented at these general sessions, most of them dealing with relations between mathematics and other sciences. About an equal number of expository papers on various mathematical topics were given. In addition, thirty-nine papers of a research nature were presented at the sectional meetings.

There were many pleasant social features, including lunches, dinners, dances, and a one-day trip to Monterrey.

One important outcome of the Congress was the decision to establish a Mexican Mathematical Society, which will sponsor a journal. The Permanent Commission of the Congress, which will have charge of the organization of the Society, is composed of Dr. Alfonso Nápoles Gándara, President; Ing. Francisco José Alvarez, Secretary; Dr. Manuel Sandoval Vallarta; Dr. Carlos Graef Fernández; Professor Alberto Barajas Celis.

P. R. RIDER

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

FROM SELECTIVE SERVICE

Occupational Bulletins No. 10 and No. 23 were amended December 14, 1942. The amendment of particular significance to mathematicians pertains to the occupational classification of graduate students. The statement follows.

"A graduate or postgraduate student undertaking further studies in these scientific and specialized fields following completion of his normal undergraduate course of study may be considered for occupational classification if, in addition to pursuing further studies, he is also acting as a graduate assistant in a recognized university or college. A graduate assistant is a student who in addition to pursuing such further studies is engaged in one of the following:

(a) In scientific research certified by a recognized federal agency as related to the war effort; or

(b) in classroom or laboratory instruction for not less than twelve hours per week."

FROM THE WAR DEPARTMENT

A memorandum entitled, "Call to Active Duty of Students Enlisted in the Enlisted Reserve Corps, Unassigned Group," was issued by the War Department upon January 27, 1943. This memorandum indicates that after the current academic period many students in the Enlisted Reserve Corps and in other categories will be detailed for instruction under the Army Specialized Training Program. Thus, the following statement from the same bulletin will be of interest to mathematicians.

"The following fields of training listed in Selective Service Bulletin No. 10 (amended December 14, 1942) as 'Critical Occupations' are accepted under the Army Specialized Training Program as approved technical engineering courses:

(1) Aeronautical engineers
(2) Automotive engineers
(3) Chemical engineers
(4) Civil engineers
(5) Electrical engineers
(6) Heating, ventilating, refrigerating, and air-conditioning engineers
(7) Mechanical engineers
(8) Radio engineers
(9) Chemists
(10) Mathematicians
(11) Meteorologists
(12) Physicists, including astronomers
(13) Psychologists"

THE PROBLEM OF SECURING TEACHERS OF COLLEGIATE MATHEMATICS FOR WARTIME NEEDS

The eighteen-year-old draft law and the inauguration of the proposed Army and Navy programs for colleges and universities are creating serious personnel problems for departments of mathematics. In an attempt to alleviate this situation, partially at least, the American Mathematical Society and the Mathematical Association of America are establishing in the Philadelphia office of the Society a bureau to provide information regarding available teachers of collegiate mathematics. The cooperation of all mathematicians is needed in order to make this plan a success.

The government has indicated that approximately 250,000 trainees will be sent to a selected group of about 300 colleges and universities. It is estimated that the full-time services of at least 2,500 teachers of mathematics will be required for this program. In addition, institutions will necessarily provide a normal program for men under 18, for women, who are electing mathematics in increasing numbers, and for students rejected, for any reason, by Selective Service. Institutions will also continue special contracts already in existence such as meteorology and aviation pre-flight schools.

To provide for all this instruction there are available about 3,000 persons now functioning as teachers of mathematics in four-year colleges. To ensure the proper handling of this instruction, readjustments of and additions to teaching staffs of many institutions will be necessary. For some institutions, additional personnel might be secured from the following sources:

(1) Members of departments of mathematics in institutions which do not assume government programs;

(2) Persons now teaching in non-critical fields who have had sufficient training in mathematics to make them valuable in this field. These persons should be recruited for the teaching of mathematics rather than being allowed to assume non-academic employment. This should probably be the main source of supply.

It is in order to promote an orderly solution of these problems that the American Mathematical Society and the Mathematical Association of America have established the bureau of information regarding teachers of collegiate mathematics. The purpose is two-fold:

(1) To collect information concerning persons who are trained in mathematics and who are or will be available for teaching during the emergency at institutions other than their own. It is requested that the names and present addresses of persons who are available for employment as teachers of collegiate mathematics be reported to the address given below. Additional information concerning qualifications will be secured directly from the mathematicians so reported.

(2) To furnish to chairmen of departments of mathematics who need additional personnel the information obtained through (1). As the employment offered will, in most cases, be temporary, an attempt will be made to furnish the chairman with an adequate list, mainly of persons living in his vicinity. *The*

purpose of the bureau is to furnish only information and not recommendations. The department chairman will, of course, investigate on his own responsibility the qualifications of the available teachers.

The country's awakened consciousness of the need for mathematics as a preparation for service in the armed forces imposes on the mathematical profession the grave responsibility to meet this need by providing the best possible teaching. The Society and Association aim to make a definite contribution to the war effort by supplying information regarding available teachers of mathematics. To do this, the full cooperation of all mathematicians is needed. It is impossible to predict what portion of the demand will be met in this manner. It is likely that teachers will become available less rapidly than the demands for additional personnel are made, but the cooperation of mathematicians will enable the bureau to function effectively.

All correspondence in this connection should be addressed to:

Committee on Available Teachers of Collegiate Mathematics
110 Bennett Hall
University of Pennsylvania
Philadelphia, Pa.

March, 1943

Committee:
W. D. CAIRNS
ARNOLD DRESDEN
J. R. KLINE

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11-13, 1943.

Twenty-seventh Annual Meeting, Cleveland, Ohio.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 17, 1943
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN, Denver, April, 1943
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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VOLUME 50 NUMBER 4

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